Fast, Accurate and Provable Triangle Counting in Fully Dynamic Graph Streams

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Given a stream of edge additions and deletions, how can we estimate the count of triangles in it? If we can store only a subset of the edges, how can we obtain unbiased estimates with small variances?

Counting triangles (i.e., cliques of size three) in a graph is a classical problem with applications in a wide range of research areas, including social network analysis, data mining, and databases. Recently, streaming algorithms for triangle counting have been extensively studied since they can naturally be used for large dynamic graphs. However, existing algorithms cannot handle edge deletions or suffer from low accuracy.

Can we handle edge deletions while achieving high accuracy? We propose THINKD, which accurately estimates the counts of global triangles (i.e., all triangles) and local triangles associated with each node in a fully dynamic graph stream with additions and deletions of edges. Compared to its best competitors, THINKD is **(a) Accurate:** up to *4.3× more accurate* within the same memory budget, **(b) Fast:** up to *2.2× faster* for the same accuracy requirements, and **(c) Theoretically sound:** always maintaining estimates with zero bias (i.e., the difference between the true triangle count and the expected value of its estimate) and small variance. As an application, we use THINKD to detect suddenly emerging dense subgraphs, and we show its advantages over state-of-the-art methods.

CCS Concepts: •**Information systems** → **Data mining;** •**Theory of computation** → **Dynamic graph algorithms; Sketching and sampling;**

Additional Key Words and Phrases: Triangle Counting, Local Triangles, Edge Deletions

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1. INTRODUCTION

Given a fully dynamic graph stream with edge additions and deletions, how can we accurately estimate the count of triangles in it with fixed memory size?

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The count of triangles (i.e., cliques of size three) is a key primitive in graph analysis with a wide range of applications, including spam/anomaly detection [Becchetti et al. 2010; Lim et al. 2018], link recommendation [Epasto et al. 2015b; Tsourakakis et al. 2011], community detection [Berry et al. 2011], degeneracy estimation [Shin et al. 2018a], and query optimization [Bar-Yossef et al. 2002]. In particular, many important metrics in social network analysis, including the clustering coefficient [Watts and Strogatz 1998], the transitivity ratio [Newman 2003], and the triangle connectivity [Batagelj and Zaveršnik 2007], are based on the count of triangles. We refer interested readers to [Hu et al. 2014; Chu and Cheng 2012; Kolountzakis et al. 2010; Suri and Vassilvitskii 2011] for detailed descriptions of the applications.

Many real graphs are best represented as a sequence of edge additions and deletions, and they often need to be processed in real time. For example, many social networking service companies aim to detect fraud or spam as quickly as possible in their online social networks, which evolve indefinitely with both edge additions and deletions. Another example is to examine graphs of data traffic and improve the network performance in real time.

As a result, there has been great interest in streaming algorithms, which gradually update their outputs as each edge insertion or deletion is received rather than operating on the entire graph at once. However, existing streaming algorithms for triangle counting focus on insertion-only streams [Ahmed et al. 2017; Jha et al. 2013; Lim et al. 2018; Pavan et al. 2013; Tangwongsan et al. 2013; Shin et al. 2018b] or greatly sacrifice accuracy to support edge deletions [De Stefani et al. 2017; Han and Sethu 2017; Kutzkov and Pagh 2014].

Why is it challenging to accurately estimate the count of triangles while handling deletions? Most existing streaming algorithms for triangle counting are randomized algorithms, whose outputs are random variables. Their accuracy depends on the bias and variance of the random variables, which are estimates of the triangle counts. State-of-the-art algorithms [De Stefani et al. 2017; Shin 2017; Lim et al. 2018; Shin et al. 2018b] discover a subset of triangles, within a fixed memory budget, and obtain unbiased estimates[1] by dividing the size of the subset by the probability that each triangle is discovered during the process. To obtain unbiased estimates, the discovery probability needs to be computed exactly, while to reduce variances, the discovery probability needs to be increased. Instead of using only the edges stored in memory, the aforementioned algorithms, which are for triangle counting in insertion-only streams, uses every arrived edge to improve its estimation, even if the edge is about to be discarded without being stored. While this idea of utilizing edges to be discarded significantly increases the discovery probability and thus reduces the variance of estimates, it has not been applied to triangle counting in graph streams with edge deletions, for which thus state-of-the-art algorithms [De Stefani et al. 2017] suffer from low accuracy. A major challenge of the application is to compute the exact discovery probability, which is necessary to obtain unbiased estimates.

Can we accurately estimate the count of triangles in fully dynamic graph streams with edge additions and deletions? In this work,[2] we propose THINKD (**Think** before you **D**iscard), an accurate streaming algorithm for global and local triangle counting in such a graph stream. That is, THINKD maintains and updates estimates of the counts of global triangles (i.e., all triangles) and local triangles incident to each node. For this problem, THINKD is the first algorithm that utilizes edges to be discarded. We first de-

[1]That is, the expected values of the estimates are equal to the true triangle counts.

[2]This work is an extended version of [Shin et al. 2018] with proofs, a variance analysis, additional experimental results, an extension of THINKD for multigraph streams, and an application of THINKD to detecting suddenly emerging dense subgraphs.

rive formulas of the exact probability that each added or deleted triangle is discovered when edges to be discarded are used with two different sampling schemes: Bernoulli trials and Random Pairing (RP) [Gemulla et al. 2008]. Then, based on the formulas, THINKD utilizes edges to be discarded while guaranteeing the unbiasedness of its estimates. We formally prove that THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$, which are two versions of THINKD based on Bernoulli trials and RP, respectively, produce unbiased estimates. We also prove a formula of the variance of estimates given by THINKD$_{\text{FAST}}$; and the time and space complexities of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$. Additionally, through extensive experiments using 8 real-world graphs, we demonstrate that the estimates have smaller variances and thus smaller errors than estimates obtained by state-of-the-art competitors, which fail to utilize edges to be discarded. Specifically, THINKD has the following strengths:

— **Accurate**: THINKD gives up to $4\times$ and $4.3\times$ *smaller estimation errors* for global and local triangle counts, respectively, than its best competitors within the same memory budget (Fig. 4).
— **Fast**: THINKD *scales linearly* with the size of the input stream (Fig. 3, Corollary 1, and Theorem 4). Especially, THINKD is up to $2.2\times$ *faster* than its best competitors with similar accuracies.
— **Theoretically Sound**: We prove the formulas for the bias and variance of the estimates given by THINKD (Theorems 1 and 2). In particular, we show that THINKD always maintains *unbiased* estimates with small variances (Fig. 2).

Additionally, as a new application, we use THINKD to detect suddenly emerging dense subgraphs, and we show its advantages over state-of-the-art methods.

Reproducibility: The source code and datasets used in the paper are available at http://dmlab.kaist.ac.kr/~kijungs/codes/thinkd/.

The rest of the paper is organized as follows. In Sect. 2, we review related work. In Sect. 3, we introduce notations and the problem definition. In Sect. 4, we describe our proposed algorithm THINKD and analyze its accuracy and complexity. After providing experimental results in Sect. 5, we conclude in Sect. 6. In Appendix A, we present a toy example that highlights (dis)advantages of THINKD and its competitors. In Appendix B, we present a theoretical analysis of the variances of estimates given by THINKD. In Appendix C, we extend THINKD to triangle counting in fully dynamic multigraph streams with parallel edges. In Appendix D, we apply THINKD to the task of detecting suddenly emerging dense subgraphs.

2. RELATED WORK

We review previous work on triangle counting in insertion-only or fully-dynamic graph streams. See Table I for a comparison of streaming algorithms for triangle counting.

2.1. Triangle Counting in Insertion-only Graph Streams

In DOULION [Tsourakakis et al. 2009], each edge in the input graph is sampled independently with probability r. Then, the probability that all three edges in a triangle are sampled and thus discovered is r^3. Based on this fact, DOULION provides $1/r^3$ times the triangle count in the graph composed of the sampled edges as an unbiased estimate of the triangle count in the entire graph.[3] As in DOULION, once the probability that each triangle is discovered (i.e., sampled) is known, an unbiased estimate is easily obtained. Since the variance of such unbiased estimates is roughly inversely propor-

[3]The expected value of an unbiased estimate is equal to the true value.

Table I: Comparison of algorithms for triangle counting in graph streams. Note that THINKD satisfies all the criteria while clearly outperforming TRIEST$_{\text{FD}}$ (which also satisfies all the criteria) in terms of speed and accuracy.

	THINKD (Proposed)	TRIEST$_{\text{FD}}$ [De Stefani et al. 2017]	ESD [Han and Sethu 2017]	MASCOT [Lim et al. 2018]	TRIEST$_{\text{IMPR}}$ [De Stefani et al. 2017]	WRS [Shin 2017]	FURL [Jung et al. 2016]	PARTITIONCT [Wang et al. 2017]	Others*
Counting Global Triangles	✓	✓	✓	✓	✓	✓		✓	✓
Counting Local Triangles	✓	✓		✓	✓	✓	✓		
Handling Large Graphs**	✓	✓		✓	✓	✓	✓	✓	✓
Handling Edge Insertions	✓	✓	✓	✓	✓	✓	✓	✓	✓
Handling Edge Deletions	✓	✓	✓						
Handling Parallel Edges	✓	✓		✓	✓		✓	✓	
Guaranteeing Unbiasedness***	✓	✓	✓	✓	✓	✓		✓	✓

* [Ahmed et al. 2017; Ahmed et al. 2014; Tangwongsan et al. 2013; Jha et al. 2013; Pavan et al. 2013]

* graphs that do not fit in memory

** giving estimates whose expected values are equal to the true triangle counts.

tional to the probability that each triangle is discovered,[4] many studies have focused on increasing the probability to reduce such variance. COLORFUL TRIANGLE SAMPLING [Pagh and Tsourakakis 2012] increases the probability to r^2 in a non-streaming setting by (a) coloring each node in the input graph with a color chosen uniformly at random among $1/r$ colors and (b) sampling the edges connecting nodes with the same color. MASCOT [Lim et al. 2018] increases the probability to r^2 in insertion-only graph streams by (a) sampling each incoming edge independently with probability r but (b) discovering (i.e., counting) triangles with each incoming edge before sampling or discarding it. Note that, in MASCOT, each triangle is discovered if and only if the first two edges of the triangle are sampled (with probability r^2). TRIEST$_{\text{IMPR}}$ [De Stefani et al. 2017] further increases the probability in insertion-only graph streams by sampling edges using the reservoir sampling [Vitter 1985], which uniformly samples as many edges as possible within a given memory budget, while MASCOT may discard edges even when memory is underutilized. Other approaches for increasing the probability include (a) sampling wedges (i.e., paths of length two) in addition to edges [Jha et al. 2013; Pavan et al. 2013; Tangwongsan et al. 2013] and (b) sampling edges with different probabilities that depend on the counts of incident triangles and adjacent sampled edges [Ahmed et al. 2014; Ahmed et al. 2017].

In addition, regarding triangle counting in insertion-only streams, handling duplicated edges [Wang et al. 2017][5], exploiting temporal dependencies of edges [Shin 2017], utilizing a computer cluster [Shin et al. 2018b], reducing variances of estimates by combining past estimates with the current one (at the expense of losing unbiasedness)

[4]For example, in THINKD$_{\text{FAST}}$, each triangle is discovered with probability r^2 in Lemma 5 and the variance of the estimate is $O(1/r^2)$ in Theorem 2.

[5]Unlike in Appendix C, [Wang et al. 2017] aims to count triangles while ignoring duplicated edges.

[Jung et al. 2016], and semi-streaming algorithms requiring multiple passes over data [Kolountzakis et al. 2010; Tsourakakis 2008] were discussed.

2.2. Triangle Counting in Fully-dynamic Graph Streams

The first algorithm for triangle counting in fully dynamic graph streams with edge deletions was proposed in [Kutzkov and Pagh 2014]. This algorithm adapts COLORFUL TRIANGLE SAMPLING, described in the previous subsection, to obtain sparsified graphs in a fully dynamic graph stream. Then, the ratio of 2-paths that are completed to triangles is estimated from these graphs. This estimated ratio is multiplied with an estimate of the total number of 2-paths in the input graph to estimate the total number of triangles. This algorithm, however, is inapplicable to real-time applications since it expensively computes an estimate once at the end of the stream instead of always maintaining an estimate. Moreover, in the worst case, the algorithm requires more memory than what is needed to store the entire input graph, as pointed out in [De Stefani et al. 2017]. ESD [Han and Sethu 2017] maintains the current snapshot of the input graph, which is given as a fully dynamic graph stream. For each change $(\{u,v\},\pm)$ in the input graph, ESD tosses a (biased) coin. If the head comes up, ESD estimates the changes in the triangle count, instead of exactly computing them, by checking whether a random neighbor of node u (or v) is also a neighbor of node v (or u), and updates its estimate of the triangle count. Otherwise, ESD does not update its estimate of the triangle count.

Unlike the previous algorithm, ESD always maintains its estimate of the triangle count. However, its scalability is limited since ESD maintains the entire input graph in memory. TRIEST$_{\text{FD}}$ [De Stefani et al. 2017] maintains edges uniformly sampled within a given memory budget in a fully dynamic graph by employing Random Pairing [Gemulla et al. 2008] (see Sect. 4.1). TRIEST$_{\text{FD}}$ also maintains unbiased estimates of global and local triangle counts in the input graph, which are obtained by multiplying (a) the triangle counts in the graph consisting of the sampled edges and (b) the reciprocal of the probability that each triangle is sampled. While TRIEST$_{\text{FD}}$ simply discards unsampled edges without utilizing them to update its estimates, our proposed algorithm, THINKD, utilizes these unsampled edges to update estimates before discarding them to minimize information loss, and thus it obtains more accurate estimates than TRIEST$_{\text{FD}}$. Although this idea of using unsampled edges was first used in MASCOT [Lim et al. 2018] and TRIEST$_{\text{IMPR}}$ [De Stefani et al. 2017], which are described in the previous subsection, for triangle counting in insertion-only streams, applying the idea to fully dynamic graph streams has remained unexplored.

In addition, semi-streaming algorithms, which require multiple passes over data, were developed for triangle counting in fully dynamic graph streams [Becchetti et al. 2010].

3. NOTATIONS AND PROBLEM DEFINITION

In this section, we first introduce notations and concepts. Then, we formally define the problem of triangle counting in fully-dynamic graph streams.

3.1. Notations

Table II lists the symbols frequently used in the paper. Consider an undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with nodes $\mathcal{V}$ and edges $\mathcal{E}$. Each edge $\{u,v\}\in\mathcal{E}$ connects two distinct nodes $u\neq v\in\mathcal{V}$. We say a subset $\{u,v,w\}\subset\mathcal{V}$ of size 3 is a *triangle* if every pair of distinct nodes u, v, and w is connected by an edge in $\mathcal{E}$. We denote the set of *global triangles* (i.e., all triangles) in $\mathcal{G}$ by $\mathcal{T}$ and the set of *local triangles* of each node $u\in\mathcal{V}$ (i.e., all triangles containing u) by $\mathcal{T}[u]\subset\mathcal{T}$.

Table II: Table of frequently-used symbols.

	Symbol	**Definition**
Notations for Fully Dynamic Graph Streams (Sect. 3)	$e^{(t)} = (\{u,v\},\delta)$	change in the input graph $\mathcal{G}$ at time t
	$\mathcal{G}^{(t)} = (\mathcal{V}^{(t)},\mathcal{E}^{(t)})$	graph $\mathcal{G}$ at time t
	$\{u,v\}$	edge between nodes u and v
	$\{u,v,w\}$	triangle with nodes u, v, and w
	$\mathcal{T}^{(t)}$	set of global triangles in $\mathcal{G}^{(t)}$
	$\mathcal{T}^{(t)}[u]$	set of local triangles of node u in $\mathcal{G}^{(t)}$
Notations for Algorithms and Analyses (Sect. 4)	$\mathcal{S}$	set of sampled edges
	$\hat{\mathcal{N}}[u]$	set of neighbors of node u in $\mathcal{S}$
	$\bar{c}$	estimate of the count of global triangles
	$c[u]$	estimate of the count of local triangles of node u
	r	sampling probability in THINKD$_{\text{FAST}}$
	k	maximum number of sampled edges in THINKD$_{\text{ACC}}$
	$\mathcal{A}^{(t)}$	set of added triangles at time t
	$\mathcal{D}^{(t)}$	set of deleted triangles at time t

Assume the graph $\mathcal{G}$ evolves from the empty graph. We consider the *fully dynamic graph stream* representing the sequence of changes in $\mathcal{G}$, and denote the stream by $(e^{(1)},e^{(2)},...)$. For each $t \in \{1,2,...\}$, the pair $e^{(t)} = (\{u,v\},\delta)$ of an edge $\{u,v\}$ and a sign $\delta \in \{+,-\}$ denotes the change in $\mathcal{G}$ at time t. Specifically, $(\{u,v\},+)$ indicates the addition of a new edge $\{u,v\} \notin \mathcal{E}$, and $(\{u,v\},-)$ indicates the deletion of an existing edge $\{u,v\} \in \mathcal{E}$. That is, we assume that only new edges can be added (see Appendix C for triangle counting in a multigraph stream with parallel edges), and only existing edges can be deleted. We use $\mathcal{G}^{(t)} = (\mathcal{V}^{(t)},\mathcal{E}^{(t)})$ to indicate $\mathcal{G}$ at time t. That is,

$$\mathcal{E}^{(0)} = \emptyset \qquad \text{and} \qquad \mathcal{E}^{(t)} = \begin{cases} \mathcal{E}^{(t-1)} \cup \{\{u,v\}\}, & \text{if } e^{(t)} = (\{u,v\},+), \\ \mathcal{E}^{(t-1)} \setminus \{\{u,v\}\}, & \text{if } e^{(t)} = (\{u,v\},-). \end{cases}$$

Lastly, we let $\mathcal{T}^{(t)}$ denote the set of global triangles in $\mathcal{G}^{(t)}$ and $\mathcal{T}^{(t)}[u] \subset \mathcal{T}^{(t)}$ denote the set of local triangles of each node $u \in \mathcal{V}^{(t)}$ in $\mathcal{G}^{(t)}$.

3.2. Problem Definition

In this work, we address the problem of estimating the counts of global and local triangles in a fully dynamic graph stream. We assume the standard data stream model where the changes in the input stream, which may not fit in memory, can be accessed once in the given order unless they are explicitly stored in memory.

Problem 1 (Global and Local Triangle Counting in a Fully Dynamic Graph Stream)**.**

— **Given:** *a fully dynamic graph stream* $(e^{(1)},e^{(2)},...)$
(i.e., sequence of edge additions and deletions in graph $\mathcal{G}$*)*
— **Maintain:** *estimates of global triangle count* $|\mathcal{T}^{(t)}|$ *and local triangle counts* $\{(u,|\mathcal{T}^{(t)}[u]|)\}_{u\in\mathcal{V}^{(t)}}$ *of graph* $\mathcal{G}^{(t)}$ *for current* $t \in \{1,2,...\}$
— **to Minimize:** *the estimation errors.*

We follow a general approach of reducing the biases and variances of estimates simultaneously rather than minimizing a specific measure of estimation error.

4. PROPOSED METHOD: THINK BEFORE YOU DISCARD (THINKD)

We propose THINKD (**Think** before you **D**iscard), which estimates the counts of global and local triangles in a fully dynamic graph stream. For estimation with limited mem-

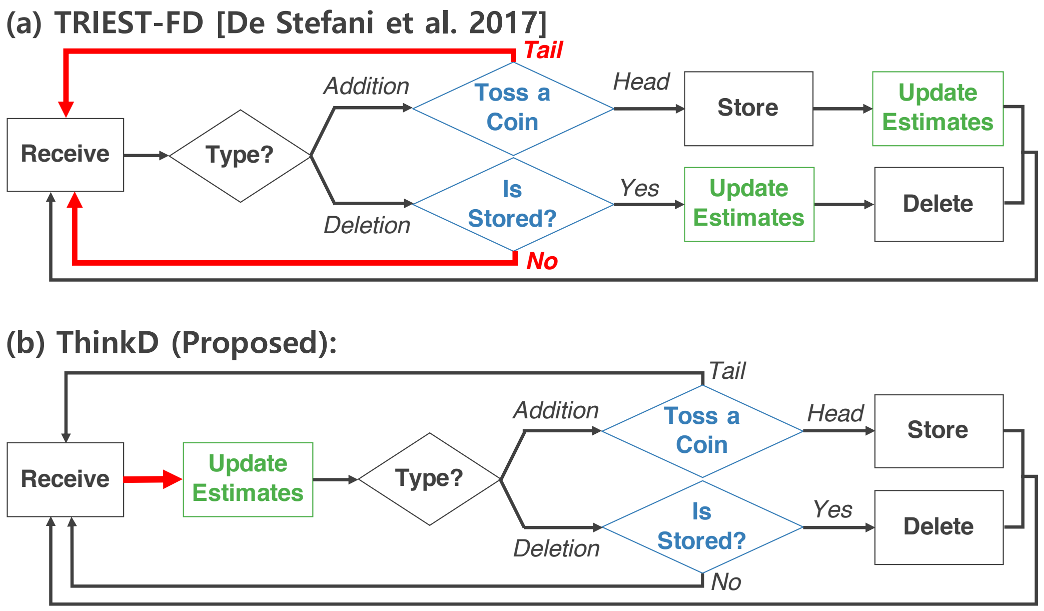


Fig. 1: Description of THINKD and its best competitor TRIEST$_{\text{FD}}$, which support global and local triangle counting in a fully dynamic graph stream. (a) In TRIEST$_{\text{FD}}$, each incoming change goes through a test step (colored blue) and then an update step (colored green). The changes not passing the test step are simply discarded (see red arrows) without being utilized to update estimates. Due to this information loss, TRIEST$_{\text{FD}}$ produces inaccurate estimates. (b) However, in THINKD (proposed), each incoming change goes through the update step (colored green) and then a test step. This guarantees that every change is used to update estimates (see the red arrow). THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ are distinguished by their testing scheme: tossing a biased coin or Random Pairing (see Sect. 4.1).

ory, THINKD samples edges and maintains those sampled edges, while discarding the other edges. The main idea of THINKD is to fully utilize unsampled edges before they are discarded, as illustrated in Fig. 1. Specifically, whenever each change in the input stream arrives, THINKD first updates its estimates using the change and previously sampled edges as follows:

— For each observed triangle addition, increase corresponding estimates by the reciprocal of the probability that the triangle addition is observed.
— For each observed triangle deletion, decrease corresponding estimates by the reciprocal of the probability that the triangle deletion is observed.

After that, if the change is an addition of an edge, THINKD decides whether to sample the edge or not.

Estimates provided by THINKD are random variables, and its estimation error depends on the bias (i.e., the difference between the true triangle count and the expected value of its estimate) and variance of the estimates. Compared to using only sampled edges, utilizing unsampled edges increases the probability that each triangle addition or deletion is observed and thus decreases the variance of estimates, which is roughly inversely proportional to the probability (see Theorem 2). Moreover, increasing or decreasing estimates by the reciprocal of the probability, for each triangle addition or deletion, makes the biases of estimates zero and thus enables THINKD provide unbiased estimates, whose expected values are equal to the true triangle counts (see Theorem 1).

In this section, we present two versions of THINKD: THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$. THINKD$_{\text{FAST}}$ uses a simple and fast edge sampling scheme based on Bernoulli trials, while THINKD$_{\text{ACC}}$ is based on Random Pairing (RP), an advanced sampling scheme for fully utilizing memory within a given budget.[6] Specifically, we focus on (a) the detailed procedures of both versions of THINKD and (b) the probabilities that each triangle is discovered during the procedures. Then, based on the probabilities, we prove the unbiasedness of both versions of THINKD and provide a variance analysis. Next, we analyze their time and space complexities. Throughout this section, we use $\bar{c}$ to denote the maintained estimate of the count of global triangles. Likewise, for each node u, we use $c[u]$ to denote the maintained estimate of the count of local triangles of node u. In addition, we let $\mathcal{S}$ be the set of currently sampled edges, and for each node u, we let $\hat{\mathcal{N}}[u]$ be the set of neighbors of u in the graph composed of the edges in $\mathcal{S}$. For each variable (e.g., $\bar{c}$), we use superscript (t) (e.g., $\bar{c}^{(t)}$) to denote the value of the variable after the t-th element $e^{(t)}$ is processed by THINKD.

4.1. Preliminaries: Random Pairing (RP)

We first introduce Random Pairing (RP) [Gemulla et al. 2008], a sampling method that THINKD is based on. Then, we prove its several properties.

RP is a uniform random sampling scheme that maintains a bounded sample size under an arbitrary sequence of insertions and deletions. The dataset $\mathcal{R}$ of interest is understood as a finite set of distinguishable items, and the sample $\mathcal{S}$ is understood as a subset of $\mathcal{R}$. The dataset $\mathcal{R}$ is initially empty, and evolves over time as items are inserted and deleted. In general, items that are deleted may be subsequently reinserted. Following [Brown and Haas 2006], a sampling scheme is called uniform if the scheme producing the sample $\mathcal{S}$ from the dataset $\mathcal{R}$ satisfies that,

$$Pr[\mathcal{S} = \mathcal{A}] = Pr[\mathcal{S} = \mathcal{B}],\ \forall \mathcal{A} \neq \mathcal{B} \subset R \text{ s.t. } |\mathcal{A}| = |\mathcal{B}|. \tag{1}$$

RP is a uniform random sampling scheme with maintaining a bounded sample size, i.e., given the sample size bound $k > 0$, $|\mathcal{S}| \leq k$ always holds.

To efficiently utilize the bounded memory space, RP keeps two counters for "uncompensated" deletions. First, the counter n_b enumerates "bad" uncompensated deletions where the sample included the deleted item so that the sample size was decremented by 1 for the deletion. Second, the counter n_g enumerates "good" uncompensated deletions where the sample excluded the deleted item so that the sample size remained the same for the deletion. Obviously, $n_b + n_g$ represents the total uncompensated deletions.

RP is described in Algorithm 1. Initially, both the dataset and the sample are empty (line 2). For the deletion, if the sample includes the deleted item, RP removes the item from the sample and increments n_b (line 13). If the sample excludes the deleted item, RP increments n_g (line 14). For the insertion, if $n_b + n_g = 0$, then there is no deletion to compensate, and the insertion is processed by Reservoir Sampling [Vitter 1985]. That is, if the sample size has not reached its upper bound (i.e., $|\mathcal{S}| < k$), RP adds the inserted item to $\mathcal{S}$ (line 5); otherwise, RP replaces a random item in $\mathcal{S}$ with the inserted item by a certain probability (lines 6-7). If $n_b + n_g > 0$, then RP tosses a coin with probability $n_b/(n_b+n_g)$ (line 8). If the head comes up with probability $n_b/(n_b+n_g)$,

[6]Using RP increases the probability that each triangle addition or deletion is observed and thus further decreases the variances of estimates. The variances are roughly inversely proportional to the probability, as stated in Theorem 2.

Algorithm 1: Random Pairing: Sampling method that THINKD uses

```
Inputs  : a dataset: $\mathcal{R}$, a sample: $\mathcal{S}$, an item to be added or deleted: $a$
Outputs: a sample after addition or deletion: $\mathcal{S}$
1  Procedure INITIALIZE():
2      $\mathcal{S} \leftarrow \emptyset$, $R \leftarrow \emptyset$, $n_b \leftarrow 0$, $n_g \leftarrow 0$
3  Procedure INSERT($a$):
4      if $n_b + n_g = 0$ then
5          if $|\mathcal{S}| < k$ then $\mathcal{S} \leftarrow \mathcal{S} \cup \{a\}$
6          else if a random number in Bernoulli($k/|R|$) is 1 then
7              replace a random item in $\mathcal{S}$ with $a$
8      else if a random number in Bernoulli($n_b/(n_b+n_g)$) is 1 then
9          $\mathcal{S} \leftarrow \mathcal{S} \cup \{a\}$, $n_b \leftarrow n_b - 1$
10     else $n_g \leftarrow n_g - 1$
11 Procedure DELETE($a$):
12     if $a \in \mathcal{S}$ then
13         $\mathcal{S} \leftarrow \mathcal{S} \setminus \{a\}$, $n_b \leftarrow n_b + 1$
14     else $n_g \leftarrow n_g + 1$
```

RP adds the inserted item to the sample and decrements n_b (line 9). If the tail comes up with probability $n_g/(n_b + n_g)$, RP decrements n_g (line 10).

In high level, RP pairs the inserted item with an uncompensated deletion if possible, and adds the inserted item to the sample if and only if the paired item was in the sample at the time of its deletion. Suppose we have $n_b + n_g > 0$ so that there are uncompensated deletions. Then for the insertion, RP pairs the inserted item with a randomly selected uncompensated deletion, called the "partner" deletion. RP adds the inserted item to the sample if and only if the partner was in the sample at the time of its deletion so that the deletion was "bad". Since the partner is chosen among n_b "bad" deletions and n_g "good" deletions, the "bad" deletion is chosen as the partner with probability $n_b/(n_b + n_g)$. To implement this, rather than tracing each partner, only the probability $n_b/(n_b + n_g)$ needs to be traced. Hence, maintaining the counters n_b and n_g suffices.

The uniformity of RP and the distribution of the sample size can be theoretically computed. For these, we let $\mathcal{R}^{(t)}$ be the dataset and $\mathcal{S}^{(t)}$ be the sample at time t. Similarly, let $n_b^{(t)}$ and $n_g^{(t)}$ be n_b and n_g at time t. Also let $y^{(t)} = \min(k, |\mathcal{R}^{(t)}| + n_b^{(t)} + n_g^{(t)})$.

The weak uniformity of RP in Lemma 1 is direct from Theorem 1 in [Gemulla et al. 2008], and we show the strong uniformity of RP in Lemma 2.

Lemma 1 (Weak Uniformity of Random Pairing). *At each fixed time t, all equal-sized subsets of the dataset have the same probability to be the set of samples maintained in Algorithm 1. Formally,*

$$Pr[\mathcal{S}^{(t)} = \mathcal{A}] = Pr[\mathcal{S}^{(t)} = \mathcal{B}],\ \forall t \geq 1,\ \forall \mathcal{A} \neq \mathcal{B} \subset \mathcal{R}^{(t)} \text{ s.t. } |\mathcal{A}| = |\mathcal{B}|. \quad (2)$$

Lemma 2 (Strong Uniformity in Random Pairing). *At each fixed time, all equal-sized subsets of the dataset have the same probability to be a subset of the samples maintained in Algorithm 1. Formally,*

$$Pr[\mathcal{A} \subset \mathcal{S}^{(t)}] = Pr[\mathcal{B} \subset \mathcal{S}^{(t)}],\ \forall t \geq 1,\ \forall \mathcal{A} \neq \mathcal{B} \subset \mathcal{R}^{(t)} \text{ s.t. } |\mathcal{A}| = |\mathcal{B}|. \quad (3)$$

Algorithm 2: THINKD$_{\text{FAST}}$: Simple and Fast Version of THINKD

```
Inputs  : fully dynamic graph stream: $(e^{(1)}, e^{(2)}, ...)$, sampling probability: $\mathcal{R}$
Outputs: estimate of the global triangle count: $\bar{c}$
         estimates of the local triangle counts: $c[u]$ for each node $u$
1  $\mathcal{S} \leftarrow \emptyset$
2  for each element $e^{(t)} = (\{u, v\}, \delta)$ in the input stream do
3      UPDATE($\{u, v\}, \delta$)
4      if $\delta = +$ then INSERT($\{u, v\}$)
5      else if $\delta = -$ then DELETE($\{u, v\}$)
6  Procedure UPDATE($\{u, v\}, \delta$):            ▷ update global and local triangle counts
7      compute $\hat{\mathcal{N}}[u] \cap \hat{\mathcal{N}}[v]$        ▷ $\hat{\mathcal{N}}[u]$ and $\hat{\mathcal{N}}[v]$ are obtained from current $\mathcal{S}$
8      for each common neighbor $w \in \hat{\mathcal{N}}[u] \cap \hat{\mathcal{N}}[v]$ do
9          if $\delta = +$ then increase $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $1/r^2$
10         else if $\delta = -$ then decrease $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $1/r^2$
11 Procedure INSERT($\{u, v\}$):
12     if a random number in Bernoulli($r$) is 1 then $\mathcal{S} \leftarrow \mathcal{S} \cup \{\{u, v\}\}$
13 Procedure DELETE($\{u, v\}$):
14     if $\{u, v\} \in \mathcal{S}$ then $\mathcal{S} \leftarrow \mathcal{S} \setminus \{\{u, v\}\}$
```

Proof. Let $e_i^{\mathcal{A}}$ be the family of size-i subsets of $\mathcal{R}^{(t)}$ including $\mathcal{A}$, and let $e_i^{\mathcal{B}}$ be the family of size-i subsets of $\mathcal{R}^{(t)}$ including $\mathcal{B}$. Then, Eq. (3) is obtained as follows:

$$\begin{aligned} Pr[\mathcal{A} \subset \mathcal{S}^{(t)}] &= \sum_i \sum_{\mathcal{C} \in e_i^{\mathcal{A}}} Pr[\mathcal{C} = \mathcal{S}^{(t)}] \\ &= \sum_i \sum_{\mathcal{C} \in e_i^{\mathcal{B}}} Pr[\mathcal{C} = \mathcal{S}^{(t)}] = Pr[\mathcal{B} \subset \mathcal{S}^{(t)}], \end{aligned}$$

where the second equality is from Eq. (2) in Lemma 1 and $|e_i^{\mathcal{A}}| = |e_i^{\mathcal{B}}|$. □

Moreover, the boundedness, the expectation, and the variance of the sample size of RP in Lemma 3 directly follow from Theorem 2 in [Gemulla et al. 2008].

Lemma 3 (Boundedness, Expectation, and Variance of the sample size of Random Pairing)**.** *At each fixed time t in Algorithm 1, the sample size always satisfies*

$$0 \leq |\mathcal{S}^{(t)}| \leq k, \ \forall t \geq 1.$$

Also, the expected value and the variance of the sample size are as follows:

$$\mathbb{E}[|\mathcal{S}^{(t)}|] = \frac{|\mathcal{R}^{(t)}| \cdot y^{(t)}}{|\mathcal{R}^{(t)}| + n_b^{(t)} + n_g^{(t)}}, \ \forall t \geq 1. \tag{4}$$

$$Var[|\mathcal{S}^{(t)}|] = \frac{(n_b^{(t)} + n_g^{(t)}) \cdot y^{(t)} \cdot (|\mathcal{R}^{(t)}| + n_b^{(t)} + n_g^{(t)} - y^{(t)}) \cdot |\mathcal{R}^{(t)}|}{(|\mathcal{R}^{(t)}| + n_b^{(t)} + n_g^{(t)})^2 \cdot (|\mathcal{R}^{(t)}| + n_b^{(t)} + n_g^{(t)} - 1)}, \forall t \geq 1. \tag{5}$$

4.2. Simple and Fast Version of THINKD: THINKD$_{\text{FAST}}$

THINKD$_{\text{FAST}}$, which is a simple and fast version of THINKD, is described in Algorithm 2. THINKD$_{\text{FAST}}$ initially has no sampled edges (line 1). Whenever each element $(\{u, v\}, \delta)$ of the input stream arrives (line 2), THINKD$_{\text{FAST}}$ first updates its estimates by calling the procedure UPDATE (line 3). Then, if the element is an addition (i.e., $\delta = +$), THINKD$_{\text{FAST}}$ samples the edge $\{u, v\}$ with a given sampling probability $\mathcal{R}$ (line 12) by calling the procedure INSERT (line 4). If the element is a deletion (i.e., $\delta = -$), THINKD$_{\text{FAST}}$ removes the edge $\{u, v\}$ from the existing samples (line 14) by calling the procedure DELETE (line 5).

In the procedure UPDATE, THINKD$_{\text{FAST}}$ finds the triangles connected by the arrived edge $\{u, v\}$ and two edges from the existing samples $\mathcal{S}$ (line 7). To this end, THINKD uses the fact that each common neighbor w of the nodes u and v in the graph composed of the sampled edges in $\mathcal{S}$ indicates the existence of such a triangle $\{u, v, w\}$. In the case of additions (i.e., $\delta = +$), since such triangles are new triangles added to the input stream, THINKD$_{\text{FAST}}$ increases the estimates of the global count and the corresponding local counts (line 9). In the case of deletions (i.e., $\delta = -$), since such triangles are those removed from the input stream, THINKD$_{\text{FAST}}$ decreases the estimates of the global count and the corresponding local counts (line 10).

As explained in detail in Sect. 4.4, for unbiased estimates, the amount of change per triangle discovered in lines 9 and 10 should be the reciprocal of the probability that each added or deleted triangle is discovered. This is because this makes the expected amount of changes in the corresponding estimates for each triangle be exactly one. In THINKD$_{\text{FAST}}$, each such triangle $\{u, v, w\}$ is discovered if and only if $\{w, u\}$ and $\{v, w\}$ are in $\mathcal{S}$, and thus the probability is r^2, as formalized in Lemma 5, which is based on Lemma 4. In both lemmas, we let $X^{(t)}$ be the random number in $Bernoulli(r)$ drawn in line 12 while the t-th element $e^{(t)}$ is processed, and we let $\mathcal{S}^{(t)}$ be $\mathcal{S}$ after the t-th element $e^{(t)}$ is processed. For each edge $\{u, v\}$, we let $l_{uv}^{(t)}$ be the last time that $\{u, v\}$ is added to or removed from $\mathcal{G}$ at time t or earlier. That is,

$$l_{uv}^{(t)} := \max(\{1 \leq s \leq t : e^{(s)} = (\{u, v\}, +) \text{ or } e^{(s)} = (\{u, v\}, -)\}). \tag{6}$$

Lemma 4. *In Algorithm 2, for each time $t \geq 1$ and any edge $\{u, v\} \in \mathcal{E}^{(t)}$, $\{u, v\} \in \mathcal{S}^{(t)}$ if and only if $X^{(l_{uv}^{(t)})} = 1$. That is,*

$$\{u, v\} \in \mathcal{S}^{(t)} \iff X^{(l_{uv}^{(t)})} = 1, \ \forall t \geq 1, \ \forall \{u, v\} \in \mathcal{E}^{(t)} \tag{7}$$

Proof. Note that $\{u, v\} \in \mathcal{E}^{(t)}$ implies that $e^{(l_{uv}^{(t)})} = (\{u, v\}, +)$, i.e. the edge $\{u, v\}$ is added at time $l_{uv}^{(t)}$. Then $\{u, v\} \notin \mathcal{E}^{(l_{uv}^{(t)}-1)}$, and since $\mathcal{S}^{(s)} \subset \mathcal{E}^{(s)}$ for all $s \geq 1$, $\{u, v\} \notin \mathcal{S}^{(l_{uv}^{(t)}-1)}$ as well. Therefore,

$$\{u, v\} \in \mathcal{S}^{(l_{uv}^{(t)})} \iff X^{(l_{uv}^{(t)})} = 1. \tag{8}$$

Also, from Eq. (6), $e^{(s)} \neq (\{u, v\}, \delta)$ if $l_{uv}^{(t)} < s \leq t$, and hence $\{u, v\}$ is not added after time $l_{uv}^{(t)}$ in Algorithm 1. Hence for all $s \in [l_{uv}^{(t)}, t]$,

$$\{u, v\} \in \mathcal{S}^{(s)} \iff \{u, v\} \in \mathcal{S}^{(l_{uv}^{(t)})}. \tag{9}$$

Combining Eq. (8) and Eq. (9) with $s = t$ gives Eq. (7). □

Lemma 5 (Discovery Probability of Triangles in THINKD$_{\text{FAST}}$). *In* THINKD$_{\text{FAST}}$*, any two distinct edges in graph $\mathcal{G}^{(t)} = (\mathcal{V}^{(t)}, \mathcal{E}^{(t)})$ are sampled with probability r^2. That is,*

$$Pr[\{u, v\} \in \mathcal{S}^{(t)} \cap \{w, x\} \in \mathcal{S}^{(t)}] = r^2, \ \forall t \geq 1, \ \forall \{u, v\} \neq \{w, x\} \in \mathcal{E}^{(t)}. \tag{10}$$

Proof. Applying Lemma 4 to the edges $\{u, v\}$ and $\{w, x\}$ gives

$$\{u, v\} \in \mathcal{S}^{(t)} \iff X^{(l_{uv}^{(t)})} = 1 \quad \text{and} \quad \{w, x\} \in \mathcal{S}^{(t)} \iff X^{(l_{wx}^{(t)})} = 1. \tag{11}$$

Then, since $X^{(s)}$'s are independent $Bernoulli(r)$ and $l_{uv}^{(t)} \neq l_{wx}^{(t)}$, applying Eq. (11) with independence of $X^{(l_{uv}^{(t)})}$ and $X^{(l_{wx}^{(t)})}$ gives

$$\begin{aligned} Pr\left[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}\right] &= Pr\left[X^{(l_{uv}^{(t)})} = 1 \cap X^{(l_{wx}^{(t)})} = 1\right] \\ &= Pr\left[X^{(l_{uv}^{(t)})} = 1\right] Pr\left[X^{(l_{wx}^{(t)})} = 1\right] = r^2. \end{aligned}$$

□

(Dis)advantages of THINKD$_{\text{FAST}}$: Due to its simplicity, THINKD$_{\text{FAST}}$ is faster than its competitors, as shown empirically in Sect. 5.4. However, it is less accurate than THINKD$_{\text{ACC}}$, described in the following subsection, since it may discard edges even when memory is not full, leading to avoidable loss of information.

4.3. Accurate Version of THINKD: THINKD$_{\text{ACC}}$

THINKD$_{\text{ACC}}$, which is an accurate version of THINKD, is described in Algorithm 3. Unlike THINKD$_{\text{FAST}}$, which may discard edges even when memory is not full, THINKD$_{\text{ACC}}$ maintains as many samples as possible within a given memory budget k (≥ 2) to minimize information loss.

To this end, THINKD$_{\text{ACC}}$ uses a sampling method called Random Pairing (RP), described in detail in Sect. 4.1. Given a fully dynamic stream with deletions, and a memory budget k, RP maintains at most k samples while satisfying the uniformity of the samples. Specifically, if we let $\mathcal{E}$ be the set of edges remaining (without being deleted) in the input stream so far and $\mathcal{S} \subset \mathcal{E}$ be the set of samples being maintained by RP, then the following equations hold, as shown in Lemma 2:

$$|\mathcal{S}| \leq k \qquad \text{and} \qquad Pr[\mathcal{A} \subset \mathcal{S}] = Pr[\mathcal{B} \subset \mathcal{S}],\ \forall \mathcal{A} \neq \mathcal{B} \subset \mathcal{E} \text{ s.t. } |\mathcal{A}| = |\mathcal{B}|.$$

This uniformity makes the probability that each triangle is discovered simple and computationally inexpensive, as shown below.

Updating the set $\mathcal{S}$ of samples using RP is described in lines 11-24. Whenever a deletion of an edge arrives, RP increases n_b or n_g depending on whether the edge is in $\mathcal{S}$ or not (lines 23 and 24). Roughly speaking, n_b and n_g denote the number of deletions that need to be "compensated" by additions (lines 17-19). If there is no deletion to compensate, RP processes each addition of an edge as in Reservoir Sampling [Vitter 1985]. That is, if memory is not full (i.e., $|\mathcal{S}| < k$), RP adds the new edge to $\mathcal{S}$ (line 14), while otherwise, RP replaces a random edge in $\mathcal{S}$ with the new edge with a certain probability (lines 15-16). See Sect. 4.1 for the definition of n_b and n_g, the details of RP, and the intuition behind the compensation; and we focus on how to use RP for triangle counting in the rest of this section.

Updating the estimates in THINKD$_{\text{ACC}}$ is the same as that in THINKD$_{\text{FAST}}$ except for the amount of change per triangle discovered in lines 9 and 10. As in THINKD$_{\text{FAST}}$, for unbiased estimates, the amount should be reciprocal of the probability that each added or deleted triangle is discovered. When each element $e^{(t)} = (\{u,v\},\delta)$ arrives, each added or deleted triangle $\{u,v,w\}$ is discovered if and only if $\{w,u\}$ and $\{v,w\}$ are in $\mathcal{S}$. As shown in Lemma 7, if we let $y = \min(k, |\mathcal{E}| + n_b + n_g)$, then the probability of such an event is

$$p(|\mathcal{E}|, n_b, n_g) := \frac{y}{|\mathcal{E}| + n_b + n_g} \times \frac{y-1}{|\mathcal{E}| + n_b + n_g - 1}. \tag{12}$$

Lemma 7 is based on Lemma 6, and in both lemmas, we let $\mathcal{E}^{(t)}$ be the set of edges remaining (without being deleted) in the input graph stream after the t-th element is processed. We also let $\mathcal{S}^{(t)} \subset \mathcal{E}^{(t)}$ and $y^{(t)}$ be $\mathcal{S}$ and y after the t-th element is processed.

Lemma 6 (Sampling Probability of Each Edge)**.** *The probability that each edge is sampled in Algorithm 3 is as follows:*

$$Pr[\{u,v\} \in \mathcal{S}^{(t)}] = \frac{y^{(t)}}{|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)}},\ \forall t \geq 1,\ \forall \{u,v\} \in \mathcal{E}^{(t)}. \tag{13}$$

Proof. Let $\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})$ be a random variable which is 1 if $\{u,v\} \in \mathcal{S}^{(t)}$ and 0 otherwise. By definition,

$$|\mathcal{S}^{(t)}| = \sum_{\{u,v\} \in \mathcal{E}^{(t)}} \mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}). \tag{14}$$

Then, by linearity of expectation and Eq. (14),

$$\mathbb{E}[|\mathcal{S}^{(t)}|] = \sum_{\{u,v\} \in \mathcal{E}^{(t)}} \mathbb{E}[\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})] = \sum_{\{u,v\} \in \mathcal{E}^{(t)}} Pr[\{u,v\} \in \mathcal{S}^{(t)}]. \tag{15}$$

Then, Eq. (13) is obtained as follows:

$$\begin{aligned} Pr[\{u,v\} \in \mathcal{S}^{(t)}] &= \frac{1}{|\mathcal{E}^{(t)}|} \sum_{\{w,x\} \in \mathcal{E}^{(t)}} Pr[\{w,x\} \in \mathcal{S}^{(t)}] \\ &= \frac{\mathbb{E}[|\mathcal{S}^{(t)}|]}{|\mathcal{E}^{(t)}|} = \frac{y^{(t)}}{|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)}}, \end{aligned}$$

where the first, second, and last equalities are from Eq. (3), Eq. (15), and Eq. (4) in Lemma 3, respectively. □

Lemma 7 (Discovery Probability of Triangles in THINKD$_{\text{ACC}}$)**.** *In* THINKD$_{\text{ACC}}$*, any two distinct edges in graph $\mathcal{G}^{(t)} = (\mathcal{V}^{(t)}, \mathcal{E}^{(t)})$ are sampled with probability as in Eq.* (12)*. That is, if we let $p^{(t)}$ and $\mathcal{S}^{(t)}$ be the values of Eq.* (12) *and $\mathcal{S}$, resp., in Algorithm 3 after the t-th element $e^{(t)}$ is processed, then*

$$Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] = p^{(t)},\ \forall t \geq 1,\ \forall \{u,v\} \neq \{w,x\} \in \mathcal{E}^{(t)}. \tag{16}$$

Proof. Let $\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})$ be a random variable which is 1 if $\{u,v\} \in \mathcal{S}^{(t)}$ and 0 otherwise. For calculating $Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}]$, we expand the covariance sum $\sum_{\{u,v\} \neq \{w,x\}} Cov(\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}), \mathbf{1}(\{w,x\} \in \mathcal{S}^{(t)}))$ in two ways and compare them.

First, we use the expansion of the variance of $\mathcal{S}^{(t)}$. From Eq. (14),

$$\begin{aligned} Var[|\mathcal{S}^{(t)}|] = &\sum_{\{u,v\} \in \mathcal{E}^{(t)}} Var[\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})] \\ &+ \sum_{\{u,v\} \neq \{w,x\}} Cov(\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}), \mathbf{1}(\{w,x\} \in \mathcal{S}^{(t)})), \end{aligned}$$

and hence the covariance sum can be expanded as

$$\begin{aligned} &\sum_{\{u,v\} \neq \{w,x\}} Cov(\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}), \mathbf{1}(\{w,x\} \in \mathcal{S}^{(t)})) \\ &= Var[|\mathcal{S}^{(t)}|] - \sum_{\{u,v\} \in \mathcal{E}^{(t)}} Var[\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})]. \end{aligned} \tag{17}$$

Algorithm 3: THINKD$_{\text{ACC}}$: Accurate Version of THINKD

```
Inputs  : fully dynamic graph stream: $(e^{(1)}, e^{(2)}, ...)$, memory budget: $k$ $(\geq 2)$
Outputs: estimate of the global triangle count: $\bar{c}$
         estimates of the local triangle counts: $c[u]$ for each node $u$
1  $\mathcal{S} \leftarrow \emptyset$, $|\mathcal{E}| \leftarrow 0$, $n_b \leftarrow 0$, $n_g \leftarrow 0$
2  for each element $e^{(t)} = (\{u,v\}, \delta)$ in the input stream do
3      UPDATE($\{u,v\}, \delta$)
4      if $\delta = +$ then INSERT($\{u,v\}$)
5      else if $\delta = -$ then DELETE($\{u,v\}$)
6  Procedure UPDATE($\{u,v\}, \delta$):              ▷ update global and local triangle counts
7      compute $\hat{\mathcal{N}}[u] \cap \hat{\mathcal{N}}[v]$      ▷ $\hat{\mathcal{N}}[u]$ and $\hat{\mathcal{N}}[v]$ are obtained from current $\mathcal{S}$
8      for each common neighbor $w \in \hat{\mathcal{N}}[u] \cap \hat{\mathcal{N}}[v]$ do
9          if $\delta = +$ then increase $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $1/p(|\mathcal{E}|, n_b, n_g)$
10         else if $\delta = -$ then decrease $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $1/p(|\mathcal{E}|, n_b, n_g)$
11 Procedure INSERT($\{u,v\}$):
12     $|\mathcal{E}| \leftarrow |\mathcal{E}| + 1$
13     if $n_b + n_g = 0$ then
14         if $|\mathcal{S}| < k$ then $\mathcal{S} \leftarrow \mathcal{S} \cup \{\{u,v\}\}$
15         else if a random number in Bernoulli($k/|\mathcal{E}|$) is 1 then
16             replace a random edge in $\mathcal{S}$ with $\{u,v\}$
17     else if a random number in Bernoulli($n_b/(n_b+n_g)$) is 1 then
18         $\mathcal{S} \leftarrow \mathcal{S} \cup \{\{u,v\}\}$, $n_b \leftarrow n_b - 1$
19     else $n_g \leftarrow n_g - 1$
20 Procedure DELETE($\{u,v\}$):
21     $|\mathcal{E}| \leftarrow |\mathcal{E}| - 1$
22     if $\{u,v\} \in \mathcal{S}$ then
23         $\mathcal{S} \leftarrow \mathcal{S} \setminus \{\{u,v\}\}$, $n_b \leftarrow n_b + 1$
24     else $n_g \leftarrow n_g + 1$
```

For the second term of Eq. (17), $Var[x] = \mathbb{E}[x^2] - (\mathbb{E}[x])^2$ implies

$$Var[\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})] = Pr[\{u,v\} \in \mathcal{S}^{(t)}] - Pr[\{u,v\} \in \mathcal{S}^{(t)}]^2. \tag{18}$$

Hence applying Eq. (13) and Eq. (18) to Eq. (17) gives the covariance sum as

$$\begin{aligned}
&\sum_{\{u,v\} \neq \{w,x\}} Cov(\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}), \mathbf{1}(\{w,x\} \in \mathcal{S}^{(t)})) \\
&= Var[|\mathcal{S}^{(t)}|] - \sum_{\{u,v\} \in \mathcal{E}^{(t)}} Var[\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)})] \\
&= Var[|\mathcal{S}^{(t)}|] - \sum_{\{u,v\} \in \mathcal{E}^{(t)}} \Big(Pr[\{u,v\} \in \mathcal{S}^{(t)}] - Pr[\{u,v\} \in \mathcal{S}^{(t)}]^2 \Big) \\
&= Var[|\mathcal{S}^{(t)}|] - |\mathcal{E}^{(t)}| \cdot \frac{y^{(t)} \cdot (|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)} - y^{(t)})}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)})^2}.
\end{aligned} \tag{19}$$

Second, we directly expand the covariance sum. Expanding the covariance sum with $Cov(x,y) = \mathbb{E}[xy] - \mathbb{E}[x] \cdot \mathbb{E}[y]$ and applying Eq. (13) give

$$\begin{aligned}
&\sum_{\{u,v\} \neq \{w,x\}} Cov(\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}), \mathbf{1}(\{w,x\} \in \mathcal{S}^{(t)})) \\
&= \sum_{\{u,v\} \neq \{w,x\}} \Big(Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] - Pr[\{u,v\} \in \mathcal{S}^{(t)}] \cdot Pr[\{w,x\} \in \mathcal{S}^{(t)}] \Big) \\
&= \sum_{\{u,v\} \neq \{w,x\}} \left(Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] - \left(\frac{y^{(t)}}{|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)}} \right)^2 \right) \\
&= \sum_{\{u,v\} \neq \{w,x\}} \Big(Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] \Big) - \frac{y^{(t)} \cdot y^{(t)} \cdot |\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)})^2}. \qquad (20)
\end{aligned}$$

Now, the probability sum $\sum_{\{u,v\} \neq \{w,x\}} Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}]$ can be obtained by comparing two expansions Eq. (19) and Eq. (20) of the covariance sum and applying Eq. (5) in Lemma 3 as

$$\begin{aligned}
&\sum_{\{u,v\} \neq \{w,x\}} Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] \\
&= \sum_{\{u,v\} \neq \{w,x\}} Cov(\mathbf{1}(\{u,v\} \in \mathcal{S}^{(t)}), \mathbf{1}(\{w,x\} \in \mathcal{S}^{(t)})) + \frac{y^{(t)} \cdot y^{(t)} \cdot |\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)})^2} \\
&= Var[|\mathcal{S}^{(t)}|] - |\mathcal{E}^{(t)}| \cdot \frac{y^{(t)} \cdot (|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)} - y^{(t)})}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)})^2} + \frac{y^{(t)} \cdot y^{(t)} \cdot |\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)})^2} \\
&= \frac{y^{(t)} \cdot (y^{(t)} - 1) \cdot |\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)}) \cdot (|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)} - 1)}. \qquad (21)
\end{aligned}$$

Then, Eq. (16) is obtained by Eq. (21) as follows:

$$\begin{aligned}
&Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] \\
&= \frac{1}{|\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)} \left(\sum_{\{u,v\} \neq \{w,x\}} Pr[\{u,v\} \in \mathcal{S}^{(t)} \cap \{w,x\} \in \mathcal{S}^{(t)}] \right) \\
&= \frac{1}{|\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)} \cdot \frac{y^{(t)} \cdot (y^{(t)} - 1) \cdot |\mathcal{E}^{(t)}| \cdot (|\mathcal{E}^{(t)}| - 1)}{(|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)}) \cdot (|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)} - 1)} \\
&= \frac{y^{(t)}}{|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)}} \cdot \frac{y^{(t)} - 1}{|\mathcal{E}^{(t)}| + n_b^{(t)} + n_g^{(t)} - 1} = p^{(t)},
\end{aligned}$$

where the first equality is from Eq. (3). □

(Dis)advantages of THINKD$_{\text{ACC}}$: Within the same memory budget, THINKD$_{\text{ACC}}$ is slower than THINKD$_{\text{FAST}}$ since THINKD$_{\text{ACC}}$ maintains and processes more samples on average. However, THINKD$_{\text{ACC}}$ is more accurate than THINKD$_{\text{FAST}}$ by utilizing more samples. These are shown empirically in Sect. 5.3 and Sect. 5.4.

Reducing estimation errors by sacrificing unbiasedness: The estimates (i.e., $\bar{c}$ and $c[u]$ for each node u) in Algorithms 2 and 3 can have negative values. Since true triangle counts are always non-negative, lower bounding the estimates by zero always reduces the estimation errors. However, the estimates become biased, and Theorem 1 in the following section does not hold anymore.

4.4. Accuracy Analyses

We prove that THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ maintain unbiased estimates, whose expected values are equal to the true global and local triangle counts. Then, we analyze the variances of the estimates that THINKD$_{\text{FAST}}$ maintains. To this end, for each variable (e.g., $\bar{c}$) in Algorithms 2 and 3, we use superscript (t) (e.g., $\bar{c}^{(t)}$) to denote the value of the variable after the t-th element $e^{(t)}$ is processed.

We first define *added triangles* and *deleted triangles* in Definitions 1 and 2.

Definition 1 (Added Triangles)**.** *Let $\mathcal{A}^{(t)}$ be the set of triangles that have been added to graph $\mathcal{G}$ at time t or earlier. Formally,*

$$\mathcal{A}^{(t)} := \{(\{u,v,w\},s) : 1 \leq s \leq t \textit{ and } \{u,v,w\} \notin \mathcal{T}^{(s-1)} \textit{ and } \{u,v,w\} \in \mathcal{T}^{(s)}\},$$

where addition time s is for distinguishing triangles composed of the same nodes but added at different times.[7]

Definition 2 (Deleted Triangles)**.** *Let $\mathcal{D}^{(t)}$ be the set of triangles that have been removed from graph $\mathcal{G}$ at time t or earlier. Formally,*

$$\mathcal{D}^{(t)} := \{(\{u,v,w\},s) : 1 \leq s \leq t \textit{ and } \{u,v,w\} \in \mathcal{T}^{(s-1)} \textit{ and } \{u,v,w\} \notin \mathcal{T}^{(s)}\},$$

where deletion time s is for distinguishing triangles composed of the same nodes but deleted at different times.[2]

Similarly, for each node $u \in \mathcal{V}^{(t)}$, we use $\mathcal{A}^{(t)}[u] \subset \mathcal{A}^{(t)}$ and $\mathcal{D}^{(t)}[u] \subset \mathcal{D}^{(t)}$ to denote the added and deleted triangles with node u, respectively. Lemma 8 formalizes the relationship between these concepts and the number of triangles.

Lemma 8 (Count of Triangles in the Current Graph)**.** *The count of triangles in the current graph equals the count of added triangles subtracted by the count of deleted triangles. Formally,*

$$|\mathcal{T}^{(t)}| = |\mathcal{A}^{(t)}| - |\mathcal{D}^{(t)}|, \ \forall t \geq 1, \tag{22}$$

$$|\mathcal{T}^{(t)}[u]| = |\mathcal{A}^{(t)}[u]| - |\mathcal{D}^{(t)}[u]|, \ \forall t \geq 1, \ \forall u \in \mathcal{V}^{(t)}. \tag{23}$$

Proof. When $t = 1$, then $\mathcal{T}^{(1)} = \mathcal{A}^{(1)} = \mathcal{D}^{(1)} = \emptyset$ holds, and hence Eq. (22) trivially holds. Hence we assume that $t \geq 2$ from now on. First, we show that for each time $s \geq 2$,

$$|\mathcal{T}^{(s)}| - |\mathcal{T}^{(s-1)}| = |\mathcal{A}^{(s)} \backslash \mathcal{A}^{(s-1)}| - |\mathcal{D}^{(s)} \backslash \mathcal{D}^{(s-1)}|. \tag{24}$$

To show this, we show the following relations,

$$|\mathcal{A}^{(s)} \backslash \mathcal{A}^{(s-1)}| = |\mathcal{T}^{(s)} \backslash \mathcal{T}^{(s-1)}|, \tag{25}$$

$$|\mathcal{D}^{(s)} \backslash \mathcal{D}^{(s-1)}| = |\mathcal{T}^{(s-1)} \backslash \mathcal{T}^{(s)}|. \tag{26}$$

For Eq. (25), note that

$$\mathcal{A}^{(s)} \backslash \mathcal{A}^{(s-1)} = \Big\{(\{u,v,w\},s) : \{u,v,w\} \notin \mathcal{T}^{(s-1)} \text{ and } \{u,v,w\} \in \mathcal{T}^{(s)}\Big\}$$

[7]Note that triangles composed of the same nodes can be added multiple times (and thus can be removed multiple times) only if deleted edges are added again.

and

$$\mathcal{T}^{(s)}\backslash\mathcal{T}^{(s-1)} = \Big\{\{u,v,w\} : \{u,v,w\} \notin \mathcal{T}^{(s-1)} \text{ and } \{u,v,w\} \in \mathcal{T}^{(s)}\Big\}$$

hold, and hence Eq. (25) holds. Similarly,

$$\mathcal{D}^{(s)}\backslash\mathcal{D}^{(s-1)} = \Big\{(\{u,v,w\},s) : \{u,v,w\} \in \mathcal{T}^{(s-1)} \text{ and } \{u,v,w\} \notin \mathcal{T}^{(s)}\Big\}$$

and

$$\mathcal{T}^{(s-1)}\backslash\mathcal{T}^{(s)} = \Big\{\{u,v,w\} : \{u,v,w\} \in \mathcal{T}^{(s-1)} \text{ and } \{u,v,w\} \notin \mathcal{T}^{(s)}\Big\}$$

hold, and hence Eq. (26) holds. Then, Eq. (25) and Eq. (26) imply

$$\begin{aligned}|\mathcal{A}^{(s)}\backslash\mathcal{A}^{(s-1)}| + |\mathcal{T}^{(s-1)}| &= |\mathcal{T}^{(s)}\backslash\mathcal{T}^{(s-1)}| + |\mathcal{T}^{(s-1)}| = |\mathcal{T}^{(s-1)} \cup \mathcal{T}^{(s)}| \\ &= |\mathcal{T}^{(s-1)}\backslash\mathcal{T}^{(s)}| + |\mathcal{T}^{(s)}| = |\mathcal{D}^{(s)}\backslash\mathcal{D}^{(s-1)}| + |\mathcal{T}^{(s)}|,\end{aligned}$$

and hence Eq. (24) holds. Then, summing up Eq. (24) from $s = 2$ to t yields

$$|\mathcal{T}^{(t)}| - |\mathcal{T}^{(1)}| = \sum_{s=2}^{t} |\mathcal{A}^{(s)}\backslash\mathcal{A}^{(s-1)}| - \sum_{s=2}^{t} |\mathcal{D}^{(s)}\backslash\mathcal{D}^{(s-1)}|. \tag{27}$$

Then, $\big\{\mathcal{A}^{(s)}\backslash\mathcal{A}^{(s-1)}\big\}_{s=2}^{t}$ being disjoint over s implies

$$\sum_{s=2}^{t} |\mathcal{A}^{(s)}\backslash\mathcal{A}^{(s-1)}| = \left|\bigcup_{s=2}^{t}(\mathcal{A}^{(s)}\backslash\mathcal{A}^{(s-1)})\right| = |\mathcal{A}^{(t)}\backslash\mathcal{A}^{(1)}|, \tag{28}$$

and similarly,

$$\sum_{s=2}^{t} |\mathcal{D}^{(s)}\backslash\mathcal{D}^{(s-1)}| = |\mathcal{D}^{(t)}\backslash\mathcal{D}^{(1)}|. \tag{29}$$

holds. Then, applying Eq. (28), Eq. (29), and $\mathcal{T}^{(1)} = \mathcal{A}^{(1)} = \mathcal{D}^{(1)} = \emptyset$ to Eq. (27) yields that for all $t \geq 2$,

$$|\mathcal{T}^{(t)}| = |\mathcal{A}^{(t)}| - |\mathcal{D}^{(t)}|,$$

which completes the proof of Eq. (22).

For Eq. (23), replacing $\mathcal{T}^{(s)}$ by $\mathcal{T}^{(s)}[u]$, $\mathcal{A}^{(s)}$ by $\mathcal{A}^{(s)}[u]$, and $\mathcal{D}^{(s)}$ by $\mathcal{D}^{(s)}[u]$ and repeating above give a proof. □

Based on these concepts, we prove that THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ maintain unbiased estimates in Theorem 1. For the unbiasedness of the estimate $\bar{c}$ of the global count, we show that the expected amount of change in $\bar{c}$ for each added triangle is $+1$, while that for each deleted triangle is -1. This follows from the fact that the amount of change in estimates for each observed triangle addition or deletion is the reciprocal of the probability that each triangle addition or deletion is observed. Then, by Lemma 8, the expected value of $\bar{c}$ equals the true global count. Likewise, we show the unbiasedness of the estimate of the local triangle count of each node by considering only the added and deleted triangles incident to the node.

Theorem 1 ('Any Time' Unbiasedness of THINKD**).** THINKD *gives unbiased estimates at any time. Formally, in Algorithms 2 and 3,*

$$\mathbb{E}[\bar{c}^{(t)}] = |\mathcal{T}^{(t)}|, \ \forall t \geq 1, \tag{30}$$

$$\mathbb{E}[c^{(t)}[u]] = |\mathcal{T}^{(t)}[u]|, \ \forall t \geq 1, \ \forall u \in \mathcal{V}^{(t)}. \tag{31}$$

Proof. Consider a triangle $(\{u, v, w\}, s) \in \mathcal{A}^{(t)}$, and let $e^{(s)} = (\{u, v\}, +)$ without loss of generality. The amount $\alpha^{(s)}_{uvw}$ of change in each of $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ due to the discovery of $(\{u, v, w\}, s)$ in line 9 of Algorithm 2 or Algorithm 3 is

$$\alpha^{(s)}_{uvw} = \begin{cases} 1/r^2 & if \{v,w\} \in \mathcal{S}^{(s-1)} \text{ and } \{w,u\} \in \mathcal{S}^{(s-1)} \text{ in Algorithm 2} \\ 1/p^{(s-1)} & if \{v,w\} \in \mathcal{S}^{(s-1)} \text{ and } \{w,u\} \in \mathcal{S}^{(s-1)} \text{ in Algorithm 3} \\ 0 & otherwise. \end{cases}$$

Then, from Eq. (10) and Eq. (16), the following equation holds:

$$\alpha^{(s)}_{uvw} = \begin{cases} \frac{1}{Pr[\{v,w\} \in \mathcal{S}^{(s-1)} \cap \{w,u\} \in \mathcal{S}^{(s-1)}]} & if \{v,w\} \in \mathcal{S}^{(s-1)} \text{ and } \{w,u\} \in \mathcal{S}^{(s-1)} \\ 0 & otherwise. \end{cases}$$

Hence,

$$\mathbb{E}[\alpha^{(s)}_{uvw}] = 1. \tag{32}$$

Consider a triangle $(\{u, v, w\}, s) \in \mathcal{D}^{(t)}$, and let $e^{(s)} = (\{u, v\}, -)$ without loss of generality. The amount $\beta^{(s)}_{uvw}$ of change in each of $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ due to the discovery of $(\{u, v, w\}, s)$ in line 10 of Algorithm 2 or Algorithm 3 is

$$\beta^{(s)}_{uvw} = \begin{cases} -1/r^2 & if \{v,w\} \in \mathcal{S}^{(s-1)} \text{ and } \{w,u\} \in \mathcal{S}^{(s-1)} \text{ in Algorithm 2} \\ -1/p^{(s-1)} & if \{v,w\} \in \mathcal{S}^{(s-1)} \text{ and } \{w,u\} \in \mathcal{S}^{(s-1)} \text{ in Algorithm 3} \\ 0 & otherwise. \end{cases}$$

Then, from Eq. (10) and Eq. (16), the following equation holds:

$$\beta^{(s)}_{uvw} = \begin{cases} \frac{-1}{Pr[\{v,w\} \in \mathcal{S}^{(s-1)} \cap \{w,u\} \in \mathcal{S}^{(s-1)}]} & if \{v,w\} \in \mathcal{S}^{(s-1)} \text{ and } \{w,u\} \in \mathcal{S}^{(s-1)} \\ 0 & otherwise. \end{cases}$$

Hence,

$$\mathbb{E}[\beta^{(s)}_{uvw}] = -1. \tag{33}$$

By definition, the following holds:

$$\bar{c}^{(t)} = \sum_{(\{u,v,w\},s) \in \mathcal{A}^{(t)}} \alpha^{(s)}_{uvw} + \sum_{(\{u,v,w\},s) \in \mathcal{D}^{(t)}} \beta^{(s)}_{uvw}.$$

By linearity of expectation, Eq. (32), Eq. (33), and Lemma 8, the following holds:

$$\begin{aligned} \mathbb{E}[\bar{c}^{(t)}] &= \sum_{(\{u,v,w\},s) \in \mathcal{A}^{(t)}} \mathbb{E}[\alpha^{(s)}_{uvw}] + \sum_{(\{u,v,w\},s) \in \mathcal{D}^{(t)}} \mathbb{E}[\beta^{(s)}_{uvw}] \\ &= \sum_{(\{u,v,w\},s) \in \mathcal{A}^{(t)}} 1 + \sum_{(\{u,v,w\},s) \in \mathcal{D}^{(t)}} (-1) = |\mathcal{A}^{(t)}| - |\mathcal{D}^{(t)}| = |\mathcal{T}^{(t)}|. \end{aligned}$$

Likewise, for each node $u \in \mathcal{V}^{(t)}$, the following holds:

$$c^{(t)}[u] = \sum_{(\{u,v,w\},s) \in \mathcal{A}^{(t)}[u]} \alpha^{(s)}_{uvw} + \sum_{(\{u,v,w\},s) \in \mathcal{D}^{(t)}[u]} \beta^{(s)}_{uvw}.$$

By linearity of expectation, Eq. (32), Eq. (33), and Lemma 8, the following holds:

$$\begin{aligned}\mathbb{E}[c^{(t)}[u]] &= \sum_{(\{u,v,w\},s)\in\mathcal{A}^{(t)}[u]} \mathbb{E}[\alpha^{(s)}_{uvw}] + \sum_{(\{u,v,w\},s)\in\mathcal{D}^{(t)}[u]} \mathbb{E}[\beta^{(s)}_{uvw}] \\ &= \sum_{(\{u,v,w\},s)\in\mathcal{A}^{(t)}[u]} 1 + \sum_{(\{u,v,w\},s)\in\mathcal{D}^{(t)}[u]} (-1) = |\mathcal{A}^{(t)}[u]| - |\mathcal{D}^{(t)}[u]| = |\mathcal{T}^{(t)}[u]|.\end{aligned}$$

□

In Appendix B, we prove the formulas for the variances of estimates given by THINKD$_{\text{FAST}}$. Theorem 2 is implied by the formulas. Note that r^2 in Eq. (34) and Eq. (35) is the discovery probability shown in Lemma 5.

Theorem 2 (Variance of THINKD$_{\text{FAST}}$). *Given an input graph stream, the variances of estimates maintained by* THINKD$_{\text{FAST}}$ *with the sampling probability $\mathcal{R}$ is proportional to $1/r^2$. Formally, in Algorithm 2,*

$$Var[\bar{c}^{(t)}] = O(1/r^2),\ \forall t \geq 1, \tag{34}$$

$$Var[c^{(t)}[u]] = O(1/r^2),\ \forall t \geq 1,\ \forall u \in \mathcal{V}^{(t)}. \tag{35}$$

Proof. See Theorem 5 in Appendix B. □

4.5. Complexity Analyses

We analyze the time and space complexities of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$. In our analyses, we use $\bar{\mathcal{V}}^{(t)} := \bigcup_{s=1}^{t} \mathcal{V}^{(s)}$ to denote the set of nodes that appear in the t-th or earlier elements in the input stream.

Space Complexity: To process the first t elements in the input graph stream, THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ maintain one estimate for the global triangle count and at most $|\bar{\mathcal{V}}^{(t)}|$ estimates for the local triangle counts. In addition, THINKD$_{\text{FAST}}$ maintains $|\mathcal{E}^{(t)}| \cdot r$ edges on average, while THINKD$_{\text{ACC}}$ maintains up to k edges. Thus, the average space complexities of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ are $O(|\mathcal{E}^{(t)}| \cdot r + |\bar{\mathcal{V}}^{(t)}|)$ and $O(k + |\bar{\mathcal{V}}^{(t)}|)$, respectively. The complexities become $O(|\mathcal{E}^{(t)}| \cdot r)$ and $O(k)$ when only the global triangle count needs to be estimated.

Time Complexity: We prove the average time complexity of THINKD$_{\text{FAST}}$ in Theorem 3, which implies Corollary 1, and the worst-case time complexity of THINKD$_{\text{ACC}}$ in Theorem 4. Corollary 1 and Theorem 4 state that, given a fixed memory budget k, THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ scale linearly with the number of elements in the input stream.

Theorem 3 (Time Complexity of THINKD$_{\text{FAST}}$). *Algorithm 2 takes $O(t+t^2r)$ on average to process the first t elements in the input stream.*

Proof. In Algorithm 2, the most expensive step in processing each element $e^{(s)} = (\{u,v\},\delta)$ is to intersect $\hat{\mathcal{N}}[u]$ and $\hat{\mathcal{N}}[v]$ (line 7), which takes $O(1 + \mathbb{E}[|\hat{\mathcal{N}}[u]| + |\hat{\mathcal{N}}[v]|]) = O(1 + \mathbb{E}[|\mathcal{S}|]) = O(1 + sr)$ on average. Hence, processing the first t elements takes $\sum_{s=1}^{t} O(1 + sr) = O(t + t^2 r)$ on average. □

Collorary 1 (Time Complexity of THINKD$_{\text{FAST}}$ with Fixed Memory k). *If $r = O(k/t)$ for a constant k (≥ 1), then Algorithm 2 takes $O(tk)$ on average to process the first t elements in the input stream.*

Theorem 4 (Time Complexity of THINKD$_{\text{ACC}}$). *Algorithm 3 takes $O(tk)$ to process the first t elements in the input stream.*

Table III: Summary of the real-world and synthetic graph streams used in our experiments. B: billion, M: million, K: thousand.

Name	#Nodes	#Edges	Type
Friendster [Yang and Leskovec 2015]	65.6M	1.81B	Friendship
Orkut [Mislove et al. 2007]	3.07M	117M	Friendship
Flickr [Mislove et al. 2007]	2.30M	22.8M	Friendship
Patent [Hall et al. 2001]	3.77M	16.5M	Citation
Youtube [Mislove et al. 2007]	3.22M	9.38M	Friendship
BerkStan [Leskovec et al. 2009]	685K	6.65M	Web
Facebook [Viswanath et al. 2009]	63.7K	817K	Friendship
Epinion [Massa and Avesani 2005]	132K	711K	Trust
Random ($\approx$ 800GB)*	1M	0.1B-100B	Synthetic

* We used 4 bytes, 8 bytes, and 1 bit to represent a node, an edge, and a sign in the stream, respectively. If 20 bits and 40 bits were used to represent a node and an edge, the stream would become about 500GB.

Proof. In Algorithm 3, the most expensive step in processing each element $e^{(s)} = (\{u,v\},\delta)$ is to intersect $\hat{\mathcal{N}}[u]$ and $\hat{\mathcal{N}}[v]$ (line 7), which takes $O(1+|\hat{\mathcal{N}}[u]|+|\hat{\mathcal{N}}[v]|) = O(k)$. Thus, processing the first t elements takes $O(tk)$. □

5. EXPERIMENTS

In this section, we review our experiments for answering the following questions:

— **Q1. Illustration of Theorems**: Does THINKD give unbiased estimates? Does THINKD scale linearly with the size of the input stream?
— **Q2. Accuracy**: Is THINKD more accurate than its best competitors?
— **Q3. Speed**: Is THINKD faster than its best competitors?
— **Q4. Effects of Deletions**: Is THINKD consistently accurate regardless of the ratio of deleted edges?
— **Q5. Application to Social Network Analysis:** Is THINKD useful for accurate, space-efficient, and incremental estimation of measures that are widely used in social network analysis and network science?

Additionally, in appendices, our experiments for answering the following questions are reviewed:

— **Q6. Speed and Accuracy in Multigraph Streams (Appendix C)**: Is a variant of THINKD for triangle counting in multigraph streams (i.e., graph streams with parallel edges) faster and more accurate than its best competitors?
— **Q7. Application to Anomaly Detection (Appendix D)**: Does THINKD-SPOT, which is based on THINKD, detect suddenly emerging dense subgraphs faster and more accurately than its best competitors?

5.1. Experimental Settings

Machines: We used a PC with a 3.60GHz Intel i7-4790 CPU and 32GB RAM unless otherwise stated.
Datasets: We created fully dynamic graph streams with deletions using the real-world graphs listed in Table III as follows: (a) create the additions of the edges in the input

graph and randomly shuffle them[8], (b) choose $\alpha\%$ of the edges and create the deletions of them, (c) locate each deletion in a random position after the corresponding addition. We set α to 20% unless otherwise stated (see Sect. 5.5 for its effect on accuracy). The created streams were streamed from the disk.
Implementations: We implemented THINKD$_{\text{FAST}}$ (Sect. 4.2), THINKD$_{\text{ACC}}$ (Sect. 4.3), TRIEST$_{\text{FD}}$ [De Stefani et al. 2017], TRIEST$_{\text{IMPR}}$ [De Stefani et al. 2017], ESD [Han and Sethu 2017], and MASCOT [Lim et al. 2018] commonly in Java. In all of them, sampled edges are stored in the adjacency list format, and as described in the last paragraph of Sect. 4.3, estimates are lower bounded by zero unless otherwise stated.
Evaluation Metrics: Let x and $\{(u, x[u])\}_{u\in\mathcal{V}}$ be the true counts of global triangles and local triangles at the end of the input stream. Let $\hat{x}$ and $\{(u, \hat{x}[u])\}_{u\in\mathcal{V}}$ be the corresponding estimates obtained by the evaluated algorithm. Then, we measured the accuracy of each algorithm using the following metrics:

— *Global Error* (the lower the better): $|x - \hat{x}|/x$,
— *RMSE* (the lower the better): $\sqrt{\frac{1}{|\mathcal{V}|}\sum_{u\in\mathcal{V}}(x[u] - \hat{x}[u])^2}$,
— *Rank Correlation* (the higher the better): Spearman's rank correlation coefficient [Spearman 1904] between $\{(u, x[u])\}_{u\in\mathcal{V}}$ and $\{(u, \hat{x}[u])\}_{u\in\mathcal{V}}$.

5.2. Q1. Illustration of Theorems

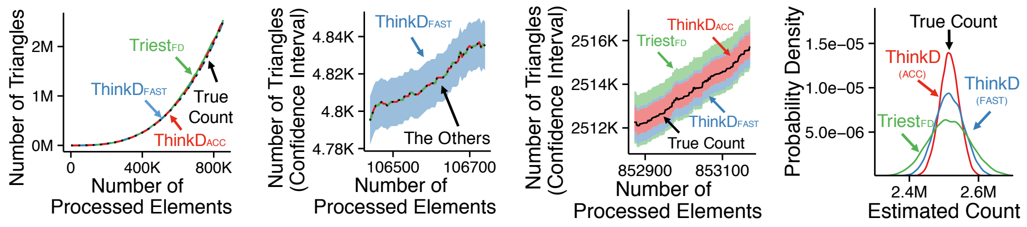
THINKD gives unbiased estimates (Theorem 1). We compared $10,000$ estimates of the global triangle count obtained by THINKD$_{\text{FAST}}$, THINKD$_{\text{ACC}}$, and TRIEST$_{\text{FD}}$, whose parameters were set so that on average 10% of the edges are stored at the end of each graph stream. Figs. 2(d) and 2(h) show the distributions of the estimates at the end of the Facebook and Epinion datasets, respectively. The means of the estimates were close to the true triangle count, consistently with Theorem 1 (i.e., unbiasedness of THINKD). Moreover, THINKD$_{\text{ACC}}$ and THINKD$_{\text{FAST}}$ gave estimates with smaller variances than TRIEST$_{\text{FD}}$. Figs. 2(b), 2(c), 2(f), and 2(g) show how the 95% confidence intervals, estimated from $10,000$ trials, changed over early and late parts of the datasets. THINKD$_{\text{ACC}}$ was consistently the most accurate with the smallest confidence intervals. THINKD$_{\text{FAST}}$ became more accurate than TRIEST$_{\text{FD}}$ as the input graph grew over time. The same trend can be seen in Figs. 2(a) and 2(e), where we show how an estimate from a single trial of each algorithm changed over the entire datasets.
THINKD scales linearly (Corollary 1 and Theorem 4). We measured the elapsed times taken by THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ to process all elements in graph streams with different numbers of elements. To measure their speeds independently of the speed of the input stream, we ignored time taken to wait for the arrival of elements. In both algorithms, we set k and r so that on average 10^7 edges are stored at the end of each input stream. Fig. 3(a) shows the results in the Random datasets, which were created by the Erdös-Rényi model. Both THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ scaled linearly with the number of elements, as expected in Corollary 1 and Theorem 4. Notice that the largest dataset contains **100** billion elements. As seen in Fig. 3(b) and Fig. 3(c), we obtained similar trends in graph streams with realistic structures created by sampling different numbers of elements from the Friendster and Orkut datasets.

5.3. Q2. Accuracy (THINKD is more accurate than its competitors)

We compared the accuracies of four algorithms that support edge deletions. As we changed the ratio of stored edges at the end of each input stream from 5% to 40%,

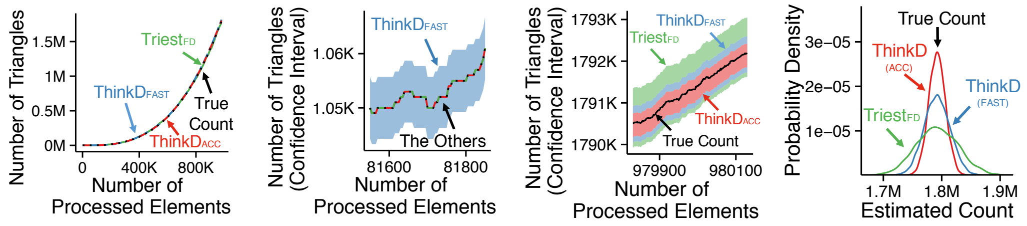
[8]Instead, additions can be ordered chronologically when timestamps of edges are given. We obtained consistent experimental results when we ordered additions randomly and chronologically.

Epinion Dataset:



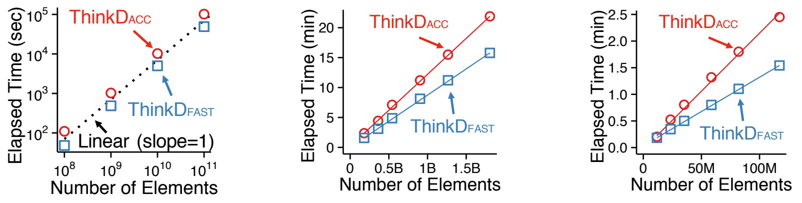
(a) Estimates over time (the entire stream)
(b) 95% confidence intervals over time (an early part of the steram)
(c) 95% confidence intervals over time (a late part of the stream)
(d) Distribution of estimates (at the end of the steram)

Facebook Dataset:



(e) Estimates over time (the entire stream)
(f) 95% confidence intervals over time (an early part of the steram)
(g) 95% confidence intervals over time (a late part of the stream)
(h) Distribution of estimates (at the end of the steram)

Fig. 2: **THINKD is provably accurate and scalable.** (a, e) THINKD is 'any time', maintaining estimates while the input graph evolves. (b-c, f-g) THINKD$_{\text{ACC}}$ is the most accurate with the smallest confidence intervals over the entire stream. THINKD$_{\text{FAST}}$ becomes more accurate than TRIEST$_{\text{FD}}$ as the input graph grows over time. (d, h) THINKD provides unbiased estimates, whose expected values are equal to the true triangle counts, and their variances are small.



(a) Random (log-log scale) (b) Friendster (c) Orkut

Fig. 3: **THINKD is scalable.** THINKD scales linearly with the size of the input stream.

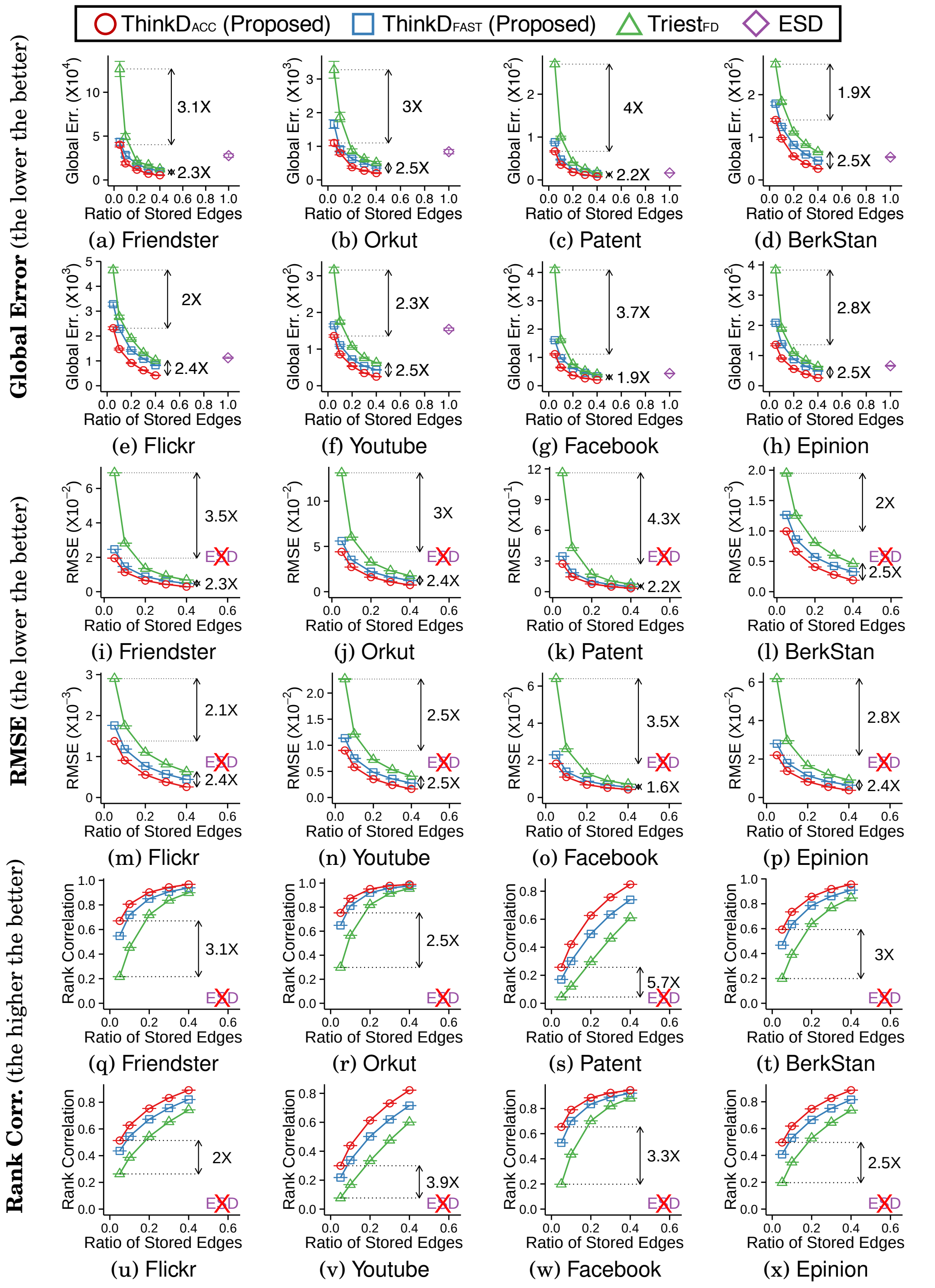


Fig. 4: **THINKD is accurate**. THINKD provides the best trade-off between space and accuracy. In particular, THINKD$_{\text{ACC}}$ is up to 4.3× **more accurate** than TRIEST$_{\text{FD}}$ within the same memory budget. Error bars denote ±1 standard error. ESD is inapplicable to local triangle counting.

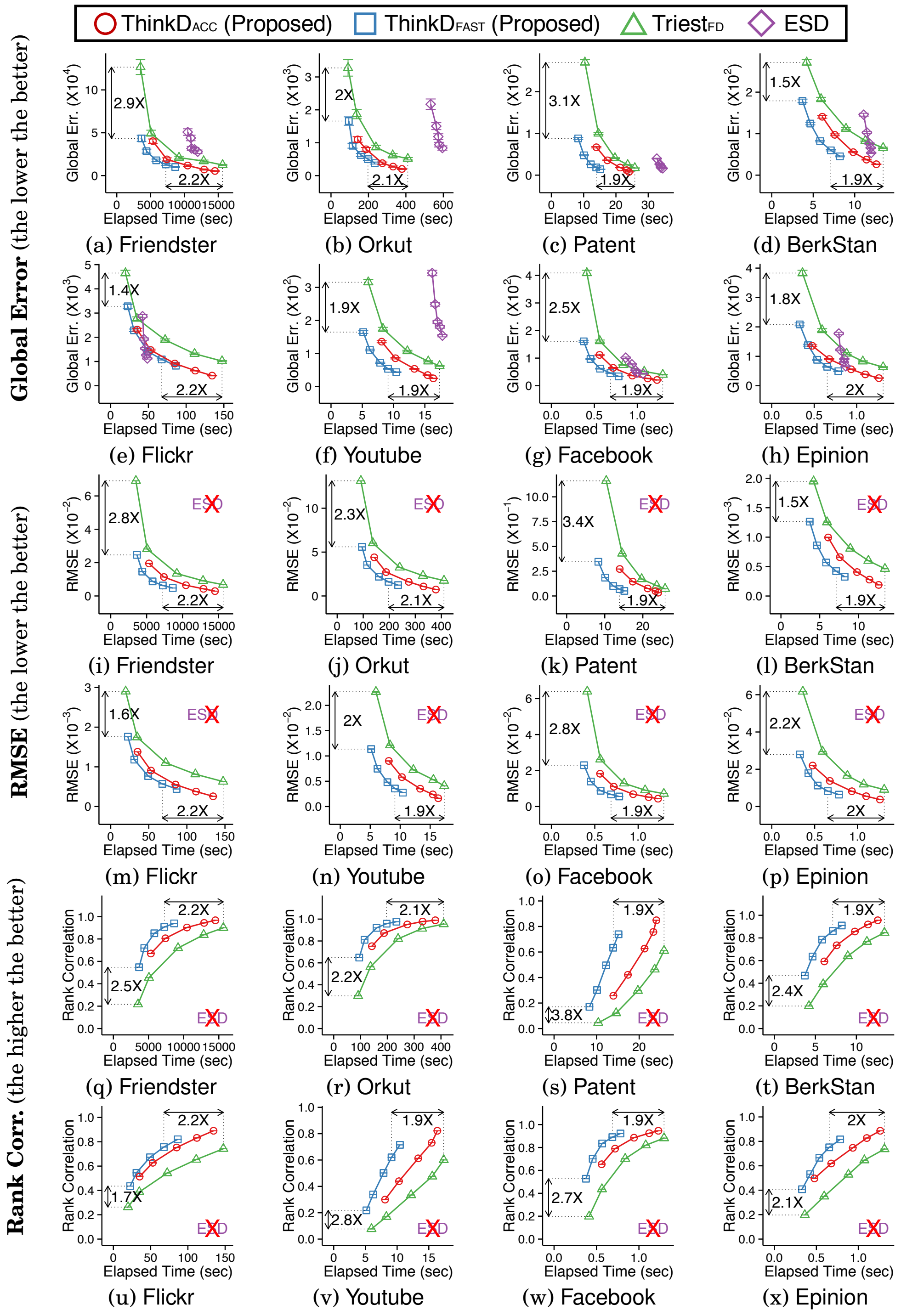


Fig. 5: **THINKD is fast**. THINKD provides the best trade-off between speed and accuracy. In particular, THINKD$_{\text{FAST}}$ is up to **2.2**$\times$ **faster** than TRIEST$_{\text{FD}}$ when they are similarly accurate. Error bars denote ± 1 standard error. ESD is inapplicable to local triangle counting.

curate than TRIEST$_{\text{FD}}$ in terms of global error and RMSE, respectively. Between our algorithms, THINKD$_{\text{ACC}}$ consistently outperformed THINKD$_{\text{FAST}}$ in terms of accuracy.

5.4. Q3. Speed (THINKD is faster than its competitors)

We compared the speeds and accuracies of four algorithms that support edge deletions. The detailed settings were the same as those in Sect. 5.3 except that we measured the performance of ESD as we changed its parameter from 0.2 to 1.0. To measure the speeds of the algorithms independently of the speed of the input stream, we ignored time taken to wait for the arrival of elements. As seen in Fig. 5, THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ consistently gave the best trade-off between speed and accuracy. Specifically, for the same global error and RMSE, THINKD$_{\text{FAST}}$ was up to $\mathbf{2.2\times}$ **faster** than TRIEST$_{\text{FD}}$. Between our algorithms, THINKD$_{\text{FAST}}$ consistently outperformed THINKD$_{\text{ACC}}$ in terms of speed.

5.5. Q4. Effects of Deletions (THINKD is consistently accurate)

We measured how the ratio of deleted edges (i.e., α in Sect. 5.1) in input graph streams affects the accuracies of the considered algorithms. In every algorithm, we set the ratio of stored edges at the end of each input stream to 10%. As seen in Fig. 6, all algorithms that support edge deletions became more accurate as input graphs became smaller with more deletions. THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ were similarly accurate with MASCOT and TRIEST$_{\text{IMPR}}$, respectively, in the streams without deletions. In the streams with deletions, which MASCOT and TRIEST$_{\text{IMPR}}$ cannot handle, THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ were $\mathbf{1.8 - 3.4\times}$ **more accurate** than TRIEST$_{\text{FD}}$ regardless of the ratio of deleted edges.

5.6. Q5. Application to Social Network Analysis

We show that THINKD can be used for accurate and space-efficient estimation of the following measures, which are widely used in social network analysis and network science [Luce and Perry 1949; Holland and Leinhardt 1971; Wasserman and Faust 1994; Watts and Strogatz 1998; Kemper 2009], in fully-dynamic graph streams:

— **Transitivity Ratio** (TR) [Luce and Perry 1949]: The *transitivity ratio* of a graph $\mathcal{G}^{(t)}$ is defined as

$$\text{TR}(\mathcal{G}^{(t)}) := \frac{6 \times |\mathcal{T}^{(t)}|}{\sum_{v \in \mathcal{V}^{(t)}} (d^{(t)}[v] \times (d^{(t)}[v] - 1))},$$

where $d^{(t)}[v]$ is the degree of $v \in \mathcal{V}^{(t)}$ in $\mathcal{G}^{(t)}$.

— **Local Clustering Coefficients** (LCC) [Watts and Strogatz 1998]: The *local clustering coefficient* of each node $v \in \mathcal{V}^{(t)}$ in a graph $\mathcal{G}^{(t)}$ is defined as

$$\text{LCC}(\mathcal{G}^{(t)}, v) := \frac{2 \times |\mathcal{T}^{(t)}[v]|}{d^{(t)}[v] \times (d^{(t)}[v] - 1)},$$

where $d^{(t)}[v]$ is the degree of $v \in \mathcal{V}^{(t)}$ in $\mathcal{G}^{(t)}$.

— **Global Clustering Coefficient** (GCC) [Watts and Strogatz 1998]: The *global clustering coefficient* of a graph $\mathcal{G}^{(t)}$ is defined as

$$\text{GCC}(\mathcal{G}^{(t)}) := \frac{1}{|\mathcal{V}^{(t)}|} \sum_{v \in \mathcal{V}^{(t)}} \text{LCC}(\mathcal{G}^{(t)}, v).$$

Specifically, to estimate these graph measures, the exact degree of each node $v \in \mathcal{V}^{(t)}$ (i.e., $d^{(t)}[v]$) was computed incrementally while $\bar{c}^{(t)}$ and $c^{(t)}[u]$ for each node $v \in \mathcal{V}^{(t)}$ were computed in THINKD$_{\text{FAST}}$, THINKD$_{\text{ACC}}$, and TRIEST$_{\text{FD}}$. Their parameters were set

Table IV: . **THINKD is useful for social network analysis (SNA).** THINKD enables accurate and space-efficient estimation of widely-used graph measures in fully-dynamic graph streams.

	SNA Measures	Transitivity Ratio	Global Clustering Coefficients	Local Clustering Coefficients	
Datasets	**Evaluation Metrics**	Global Error*	Global Error*	RMSE*	Rank Correlation**
Facebook	TRIEST$_{\text{FD}}$	0.0051	0.019	0.63	0.41
	THINKD$_{\text{FAST}}$	0.0046	0.010	0.26	0.56
	THINKD$_{\text{ACC}}$	**0.0026**	**0.010**	**0.15**	**0.68**
Epinion	TRIEST$_{\text{FD}}$	0.0083	0.036	0.74	0.51
	THINKD$_{\text{FAST}}$	0.0064	**0.012**	0.29	0.63
	THINKD$_{\text{ACC}}$	**0.0038**	0.014	**0.17**	**0.74**
BerkStan	TRIEST$_{\text{FD}}$	0.0083	0.016	1.56	0.35
	THINKD$_{\text{FAST}}$	0.0060	0.011	0.72	0.47
	THINKD$_{\text{ACC}}$	**0.0037**	**0.007**	**0.38**	**0.58**

* the lower the better ** the higher the better

so that on average 30% of the edges are stored at the end of each input graph stream.[10] Note that incrementally computing the degrees of all nodes takes $O(t)$ time and requires $O(n)$ space.[11] Then, replacing $|\mathcal{T}^{(t)}|$ and $|\mathcal{T}^{(t)}[v]|$ in the above definitions by $\bar{c}^{(t)}$ and $c^{(t)}[v]$, respectively, we estimated the measures. In all the algorithms, the additional time taken for incrementally computing the degrees of all nodes and estimating graph measures was negligible compared to the time taken for estimating the counts of triangles. To evaluate the accuracies of the estimates obtained by different algorithms, we computed the evaluation metrics defined in Sect. 5.1 at the end of each input stream. As summarized in Table IV, where each evaluation metric was averaged over 10 trials, THINKD$_{\text{ACC}}$ and THINKD$_{\text{FAST}}$ estimated all the considered graph measures more accurately than TRIEST$_{\text{FD}}$ within the same memory budget.

6. CONCLUSION

We propose THINKD, which estimates the counts of global and local triangles in a fully dynamic graph stream with edge additions and deletions. We theoretically and empirically show that THINKD has the following advantages:

— **Accurate**: THINKD is up to *4.3× more accurate* than its best competitors within the same memory budget (Fig. 4).
— **Fast**: THINKD is up to *2.2× faster* than its best competitors with similar accuracies (Fig. 5). THINKD processes *terabyte*-scale graph streams with linear scalability (Fig. 2, Corollary 1, and Theorem 4).
— **Theoretically Sound**: THINKD maintains *unbiased* estimates (Theorem 1) with small variances (Theorem 2) *at any time* while the input graph evolves.

Additionally, in Appendix D, we apply THINKD to the task of detecting suddenly emerging dense subgraphs and show its advantages over state-of-the-art methods.

[10]When estimating global clustering coefficients, we computed $c^{(t)}[u]$ for each node $u \in \mathcal{V}^{(t)}$ without lower bounding it by zero.

[11]For each addition $(\{u, v\}, +)$, increase $d^{(t)}[u]$ and $d^{(t)}[v]$ by 1, and for each deletion $(\{u, v\}, -)$, decrease $d^{(t)}[u]$ and $d^{(t)}[v]$ by 1.

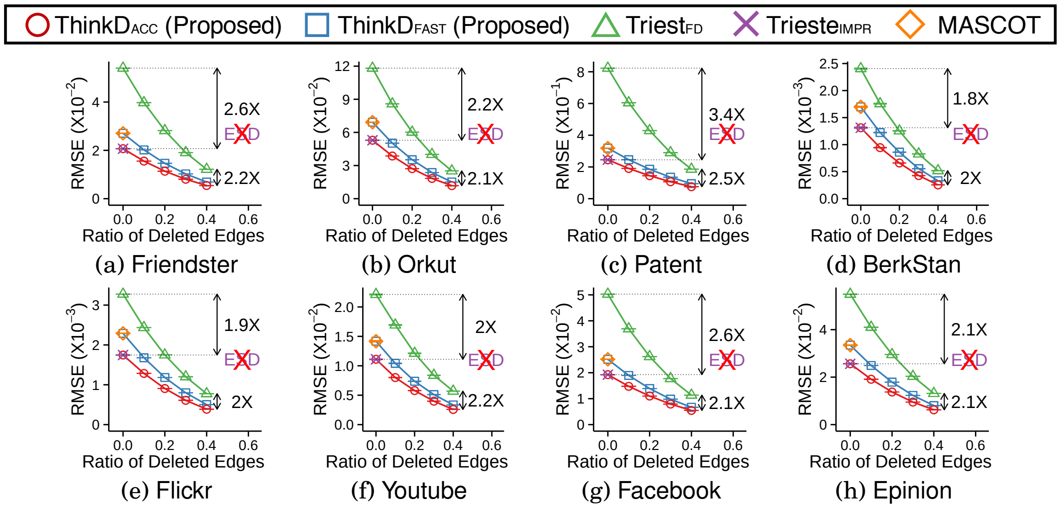


Fig. 6: **THINKD is consistently accurate regardless of the ratio of deleted edges.** Error bars denote ± 1 standard error. TRIEST$_{\text{IMPR}}$ and MASCOT are inapplicable when there are deletions. ESD is inapplicable to local triangle counting.

Reproducibility: The source code and datasets used in the paper are available at http://dmlab.kaist.ac.kr/~kijungs/codes/thinkd/.

APPENDIX

A. TOY EXAMPLE FOR THINKD AND ITS COMPETITORS (TRIEST$_{\text{FD}}$ AND ESD)

Fig. 7 shows an artificial graph stream that highlights the (dis)advantages of THINKD and its competitors (i.e., TRIEST$_{\text{FD}}$ and ESD). First, ESD runs out of memory (O.O.M.) at $t \geq 4$ since it stores all edges of the input graph. This implies that ESD has limited scalability. In the case of TRIEST$_{\text{FD}}$, its estimation at $t = 7$ is inaccurate since it does not update estimates if the current edge (i.e., $\{b, c\}$) is not stored but discarded. On the other hand, THINKD updates its estimates whenever an update (i.e., an insertion or a deletion) arrives. Hence, THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ estimate the global triangle count exactly at $t = 7$. THINKD$_{\text{FAST}}$ suffers from information loss at $t = 6$ and 7 since it discards edges (i.e., $\{c, e\}$ and $\{b, c\}$) even when the number of edges in memory is less than the given budget. On the other hand, within the given memory budget, THINKD$_{\text{ACC}}$ tries to maintain as many edges as possible in memory. Note that we set the memory budget to 3 for all methods to make a fair comparison.

B. VARIANCE ANALYSIS

We let $l_{uv}^{(t)}$ be the last time that edge $\{u, v\}$ is added to or removed from $\mathcal{G}$ at time t or earlier. For each added or deleted triangle $(\{u, v, w\}, s) \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}$, we use $\mathbf{1}_{(\{u,v,w\},s)}$ to denote the time when its first edge has arrived and $\mathbf{2}_{(\{u,v,w\},s)}$ to denote the time when its second edge has arrived. Formally,

$$\mathbf{1}_{(\{u,v,w\},s)} := \min(l_{uv}^{(s)}, l_{vw}^{(s)}, l_{wu}^{(s)}), \quad \mathbf{2}_{(\{u,v,w\},s)} := \text{median}(l_{uv}^{(s)}, l_{vw}^{(s)}, l_{wu}^{(s)}).$$

Then, we define the type of each triangle pair in Definition 3.

Time	Change	True Global Count	*ThinkD$_{Fast}$* (r = 0.5)		*ThinkD$_{ACC}$* (budget = 3)		*TRIEST$_{FD}$* (budget = 3)		*ESD* (budget = 3)	
			Sampled Edges	Estimate	Sampled Edges	Estimate	Sampled Edges	Estimate	Sampled Edges	Estimate
1	{a,b},+	0	{a,b}	0	{a,b}	0	{a,b}	0	{a,b}	0
2	{a,c},+	0	{a,b},{a,c}	0	{a,b},{a,c}	0	{a,b},{a,c}	0	{a,b},{a,c}	0
3	{a,d},+	0	{a,b},{a,c}	0	{a,b},{a,c},{a,d}	0	{a,b},{a,c},{a,d}	0	{a,b},{a,c},{a,d}	0
4	{a,e},+	0	{a,b},{a,c},{a,e}	0	{a,b},{a,c},{a,e}	0	{a,b},{a,c},{a,e}	0	O.O.M.	
5	{a,e},-	0	{a,b},{a,c}	0	{a,b},{a,c}	0	{a,b},{a,c}	0	O.O.M.	
6	{c,e},+	0	{a,b},{a,c}	0	{a,b},{a,c},{c,e}	0	{a,b},{a,c},{c,e}	0	O.O.M.	
7	{b,c},+	1	{a,b},{a,c}	1	{a,b},{a,c},{c,e}	1	{a,b},{a,c},{c,e}	0	O.O.M.	

Fig. 7: A toy example that highlights the (dis)advantages of THINKD and its competitors. ESD runs out of memory (O.O.M.) at $t \geq 4$ since it stores all edges of the input graph. TRIEST$_{\text{FD}}$ is inaccurate at $t = 7$ since it does not update estimates if an incoming edge is not stored but discarded. However, THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ produce accurate estimates although THINKD$_{\text{FAST}}$ suffers from information loss at $t = 6$ and 7.

Definition 3 (Types of Triangle Pairs). *The type of each ordered pair of two distinct triangles $\tau \neq \omega \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}$ is defined as follows:*

$$Type_{(\tau,\omega)} = \begin{cases} 1, & \textit{if } \tau \in \mathcal{A}^{(t)} \textit{ and } \omega \in \mathcal{A}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 1, \\ 2, & \textit{if } \tau \in \mathcal{D}^{(t)} \textit{ and } \omega \in \mathcal{D}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 1, \\ 3, & \textit{if } \tau \in \mathcal{A}^{(t)} \textit{ and } \omega \in \mathcal{D}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 1, \\ 4, & \textit{if } \tau \in \mathcal{D}^{(t)} \textit{ and } \omega \in \mathcal{A}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 1, \\ 5, & \textit{if } \tau \in \mathcal{A}^{(t)} \textit{ and } \omega \in \mathcal{A}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 2, \\ 6, & \textit{if } \tau \in \mathcal{D}^{(t)} \textit{ and } \omega \in \mathcal{D}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 2, \\ 7, & \textit{if } \tau \in \mathcal{A}^{(t)} \textit{ and } \omega \in \mathcal{D}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 2, \\ 8, & \textit{if } \tau \in \mathcal{D}^{(t)} \textit{ and } \omega \in \mathcal{A}^{(t)} \textit{ and } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 2, \\ 9, & \textit{otherwise (i.e., } |\{\mathbf{1}_\tau, \mathbf{2}_\tau\} \cap \{\mathbf{1}_\omega, \mathbf{2}_\omega\}| = 0). \end{cases} \tag{36}$$

Theorem 5 (Variance of THINKD$_{\text{FAST}}$). *Let $n_i^{(t)}$ be the number of Type-i triangle pairs in $\mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}$. Likewise, Let $n_i^{(t)}[u]$ be the number of Type-i triangle pairs in $\mathcal{A}^{(t)}[u] \cup \mathcal{D}^{(t)}[u]$. Then,*

$$\begin{aligned} Var[\bar{c}^{(t)}] = {} & (|\mathcal{A}^{(t)}| + |\mathcal{D}^{(t)}|) \cdot \frac{1-r^2}{r^2} \\ & + (n_1^{(t)} + n_2^{(t)} - n_3^{(t)} - n_4^{(t)}) \cdot \frac{1-r}{r} \\ & + (n_5^{(t)} + n_6^{(t)} - n_7^{(t)} - n_8^{(t)}) \cdot \frac{1-r^2}{r^2}, \ \forall t \geq 1. \end{aligned} \tag{37}$$

Likewise,

$$\begin{aligned} Var[c^{(t)}[u]] = {} & (|\mathcal{A}^{(t)}[u]| + |\mathcal{D}^{(t)}[u]|) \cdot \frac{1-r^2}{r^2} \\ & + (n_1^{(t)}[u] + n_2^{(t)}[u] - n_3^{(t)}[u] - n_4^{(t)}[u]) \cdot \frac{1-r}{r} \\ & + (n_5^{(t)}[u] + n_6^{(t)}[u] - n_7^{(t)}[u] - n_8^{(t)}[u]) \cdot \frac{1-r^2}{r^2}, \ \forall t \geq 1, \ \forall u \in \mathcal{V}^{(t)}. \end{aligned} \tag{38}$$

Proof. For each time $t \geq 1$, let $X^{(t)}$ be the random number in $Bernoulli(r)$ drawn in line 12 of Algorithm 2 while the t-th element $e^{(t)}$ is processed. Then, from Lemma 4,

$$\{u, v\} \in \mathcal{S}^{(t)} \iff X^{(l_{uv}^{(t)})} = 1. \tag{39}$$

Now, for each $\tau = (\{u, v, w\}, s) \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}$, let γ_τ be the amount of change in each of $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ due to the discovery of τ in line 9 or line 10 of Algorithm 2. Let $\delta_\tau = +1$ when $\tau \in \mathcal{A}^{(t)}$, i.e. when the last edge is added, and let $\delta_\tau = -1$ when $\tau \in \mathcal{D}^{(t)}$, i.e. when the last edge is deleted. Let $\{u, v\}$ be the edge added or deleted at time s without loss of generality. Then, $\gamma_\tau = \frac{\delta_\tau}{r^2}$ if both $\{v, w\}, \{w, u\} \in \mathcal{S}^{(s)}$ and 0 otherwise. Hence, combined with Eq. (39),

$$\gamma_\tau = \frac{\delta_\tau}{r^2} X^{(\mathbf{1}_\tau)} X^{(\mathbf{2}_\tau)}. \tag{40}$$

Then from the definitions of γ_τ, $\bar{c}^{(t)} = \sum_{\tau \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}} \gamma_\tau$, and its variance is

$$Var[\bar{c}^{(t)}] = \sum_{\tau \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}} Var[\gamma_\tau] + \sum_{\tau \neq \omega \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}} Cov[\gamma_\tau, \gamma_\omega]. \tag{41}$$

For the variance term in Eq. (41), note first that applying that $X^{(\mathbf{1}_\tau)}$ and $X^{(\mathbf{2}_\tau)}$ are independent $Bernoulli(r)$ to Eq. (40) gives $\mathbb{E}[\gamma_\tau]$ as

$$\mathbb{E}[\gamma_\tau] = \mathbb{E}\left[\frac{\delta_\tau}{r^2} X^{(\mathbf{1}_\tau)} X^{(\mathbf{2}_\tau)}\right] = \frac{\delta_\tau}{r^2} \mathbb{E}\left[X^{(\mathbf{1}_\tau)}\right] \mathbb{E}\left[X^{(\mathbf{2}_\tau)}\right] = \delta_\tau. \tag{42}$$

Then, further applying $\delta_\tau^2 = 1$ and $(X^{(s)})^2 = X^{(s)}$ to Eq. (40) and again applying that $X^{(\mathbf{1}_\tau)}$ and $X^{(\mathbf{2}_\tau)}$ are independent $Bernoulli(r)$ give $Var[\gamma_\tau]$ as

$$\begin{aligned} Var[\gamma_\tau] &= \mathbb{E}[\gamma_\tau^2] - (\mathbb{E}[\gamma_\tau])^2 = \mathbb{E}\left[\frac{\delta_\tau^2}{r^4} X^{(\mathbf{1}_\tau)} X^{(\mathbf{2}_\tau)}\right] - \delta_\tau^2 = \frac{1}{r^4} \mathbb{E}\left[X^{(\mathbf{1}_\tau)}\right] \mathbb{E}\left[X^{(\mathbf{2}_\tau)}\right] - 1 \\ &= \frac{1 - r^2}{r^2}. \end{aligned}$$

Hence the variance term in Eq. (41) is computed as

$$\sum_{\tau \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}} Var[\gamma_\tau] = \sum_{\tau \in \mathcal{A}^{(t)} \cup \mathcal{D}^{(t)}} \frac{1 - r^2}{r^2} = (|\mathcal{A}^{(t)}| + |\mathcal{D}^{(t)}|) \cdot \frac{1 - r^2}{r^2}. \tag{43}$$

For the covariance term in Eq. (41), applying Eq. (40) and Eq. (42) and using the fact that all the $X^{(s)}$'s are independent and identically distributed as $Bernoulli(r)$ and $(X^{(s)})^2 = X^{(s)}$ yield the $Cov[\gamma_\tau, \gamma_\omega]$ as

$$\begin{aligned} Cov[\gamma_\tau, \gamma_\omega] &= \mathbb{E}[\gamma_\tau \gamma_\omega] - \mathbb{E}[\gamma_\tau]\mathbb{E}[\gamma_\omega] = \mathbb{E}\left[\frac{\delta_\tau \delta_\omega}{r^4} X^{(\mathbf{1}_\tau)} X^{(\mathbf{2}_\tau)} X^{(\mathbf{1}_\omega)} X^{(\mathbf{2}_\omega)}\right] - \delta_\tau \delta_\omega \\ &= \delta_\tau \delta_\omega \left(\frac{1}{r^4} \mathbb{E}\left[\prod_{i \in \{\mathbf{1}_\gamma, \mathbf{2}_\gamma\} \cup \{\mathbf{1}_\omega, \mathbf{2}_\omega\}} X^{(i)}\right] - 1\right) \\ &= \delta_\tau \delta_\omega \left(\frac{1}{r^4} \prod_{i \in \{\mathbf{1}_\gamma, \mathbf{2}_\gamma\} \cup \{\mathbf{1}_\omega, \mathbf{2}_\omega\}} \mathbb{E}\left[X^{(i)}\right] - 1\right) = \delta_\tau \delta_\omega \left(\frac{r^{|\{\mathbf{1}_\gamma, \mathbf{2}_\gamma\} \cup \{\mathbf{1}_\omega, \mathbf{2}_\omega\}|}}{r^4} - 1\right). \end{aligned}$$

Then $\delta_\tau\delta_\omega = 1$ if $\tau,\omega \in \mathcal{A}^{(t)}$ or $\tau,\omega \in \mathcal{D}^{(t)}$, and $\delta_\tau\delta_\omega = -1$ if $\tau \in \mathcal{A}^{(t)}$, $\omega \in \mathcal{D}^{(t)}$ or $\tau \in \mathcal{D}^{(t)}$, $\omega \in \mathcal{A}^{(t)}$. Hence $Cov[\gamma_\tau, \gamma_\omega]$ can be calculated as

$$Cov[\gamma_\tau, \gamma_\omega] = \begin{cases} \frac{1-r}{r}, & \text{if } Type_{(\tau,\omega)} = 1 \text{ or } 2, \\ -\frac{1-r}{r}, & \text{if } Type_{(\tau,\omega)} = 3 \text{ or } 4, \\ \frac{1-r^2}{r^2}, & \text{if } Type_{(\tau,\omega)} = 5 \text{ or } 6, \\ -\frac{1-r^2}{r^2}, & \text{if } Type_{(\tau,\omega)} = 7 \text{ or } 8, \\ 0, & \text{if } Type_{(\tau,\omega)} = 9. \end{cases}$$

Hence the covariance term in Eq. (41) is computed as

$$\sum_{\tau\neq\omega\in\mathcal{A}^{(t)}\cup\mathcal{D}^{(t)}} Cov[\gamma_\tau, \gamma_\omega] = (n_1^{(t)} + n_2^{(t)} - n_3^{(t)} - n_4^{(t)}) \cdot \frac{1-r}{r} + (n_5^{(t)} + n_6^{(t)} - n_7^{(t)} - n_8^{(t)}) \cdot \frac{1-r^2}{r^2}. \tag{44}$$

Hence applying Eq. (43) and Eq. (44) to Eq. (41) gives

$$\begin{aligned} Var[\bar{c}^{(t)}] &= \sum_{\tau\in\mathcal{A}^{(t)}\cup\mathcal{D}^{(t)}} Var[\gamma_\tau] + \sum_{\tau\neq\omega\in\mathcal{A}^{(t)}\cup\mathcal{D}^{(t)}} Cov[\gamma_\tau, \gamma_\omega] \\ &= (|\mathcal{A}^{(t)}| + |\mathcal{D}^{(t)}|) \cdot \frac{1-r^2}{r^2} + (n_1^{(t)} + n_2^{(t)} - n_3^{(t)} - n_4^{(t)}) \cdot \frac{1-r}{r} \\ &\quad + (n_5^{(t)} + n_6^{(t)} - n_7^{(t)} - n_8^{(t)}) \cdot \frac{1-r^2}{r^2}, \end{aligned}$$

which completes the proof of Eq. (37).

For Eq. (38), replacing $\bar{c}^{(t)}$ by $c^{(t)}[u]$, $\mathcal{A}^{(t)}$ by $\mathcal{A}^{(t)}[u]$, and $\mathcal{D}^{(s)}$ by $\mathcal{D}^{(s)}[u]$ and repeating above give a proof. □

C. EXTENSIONS: TRIANGLE COUNTING IN FULLY DYNAMIC MULTIGRAPH STREAMS

C.1. Notations and Problem Definition

A *multigraph* is a graph with parallel edges; hence, two nodes can be connected by more than one edge. Specifically, we define the multiplicity of an edge $\{u, v\}$ by $M(u, v)$, which indicates a number of parallel edges between u and v; then, a multigraph has at least one edge whose $M(u, v) \geq 2$. Meanwhile, we call a graph with no parallel edges (i.e., all $M(u, v) = 1$) a *simple graph*.

Similar to the simple graph case, a *fully dynamic multigraph stream* represents the sequence of changes in a multigraph $\mathcal{G}$, and we denote the stream by $(e^{(1)}, e^{(2)}, ...)$. As in [De Stefani et al. 2017], we assume that a triple $e^{(t)} = (\{u, v\}, l, \delta)$ of an edge $\{u, v\}$, a label l, which differentiates parallel edges, and a sign $\delta \in \{+, -\}$ arrives at each time $t \in \{1, 2, ...\}$. Moreover, we assume that only new edges not in $\mathcal{G}$ (which can be parallel edges with new labels) can be added, and only existing edges in $\mathcal{G}$ can be deleted.

There are two ways of counting triangles in a multigraph. One way is to add the weights of triangles. The *weight* of each triangle $\{u, v, w\} \subset V$ is defined as $M(u, v) \times M(v, w) \times M(w, u)$. The other way is to count triangles while ignoring edge multiplicity. In this paper, we choose the former scheme, which fully reflects given information.

We address the problem of estimating the counts of global and local triangles in a fully dynamic multigraph stream, which is summarized by Problem 2. As in Sect. 3.2, we assume the standard data stream model where the elements in the input stream, which may not fit in memory, can be accessed once in the given order unless they are explicitly stored in memory.

Problem 2 (Global and Local Triangle Counting in a Fully Dynamic Multigraph Stream)**.**

— **Given:** *a fully dynamic multigraph stream with parallel edges* $(e^{(1)}, e^{(2)}, ...,)$ *(i.e., sequence of edge additions and deletions in a multigraph* $\mathcal{G}$*)*
— **Maintain:** *estimates of global triangle count* $|\mathcal{T}^{(t)}|$ *and local triangle counts* $\{(u, |\mathcal{T}^{(t)}[u]|)\}_{u\in\mathcal{V}^{(t)}}$ *of graph* $\mathcal{G}^{(t)}$ *for current* $t \in \{1, 2, ...\}$
— **to Minimize:** *the estimation errors.*

C.2. Extension of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ to a Fully Dynamic Multigraph Stream

We extend THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ to a fully dynamic multigraph stream by making two key changes while keeping the other parts the same. The overall process of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ for triangle counting in a fully dynamic multigraph stream is provided in Algorithm 4.

The first key change is to store parallel edges with their labels. Given an edge deletion, if we delete a parallel edge without considering labels, edges can be "overdeleted" and thus the counts of triangles can be underestimated. Therefore, given an edge deletion, we delete the same edge with the same label from memory (if it is in memory) without deleting parallel edges with different labels.

The second key change is the amount of changes in our estimates per new or deleted triangle. For each new observed triangle $\{u, v, w\}$, we increase the corresponding estimates by $M(v, w) \times M(w, u)/r^2$ (in THINKD$_{\text{FAST}}$) or $M(v, w) \times M(w, u)/p(|\mathcal{E}|, n_b, n_g)$ (in THINKD$_{\text{ACC}}$). We decrease the corresponding estimates by the same amounts for each observed deleted triangle $\{u, v, w\}$.

With the above proper changes, THINKD maintains unbiased estimates of the global and local triangle counts in a given fully dynamic multigraph stream. THINKD still requires sublinear memory, while the exact memory requirements depend on the sampling ratio r or the memory budget k. Note that we can apply the same techniques used in Sect. 4.4 and Sect. 4.5 to prove the accuracy and complexity of THINKD in a multigraph stream. Specifically, we can simply treat parallel edges with different labels as different edges and follow the same steps in the sections.

C.3. Experiments

We empirically validate the accuracy and speed of THINKD in order to show our extensions of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ are successful. We describe our experimental settings in Sect. C.3.1. We explain the accuracy of THINKD and other methods in Sect. C.3.2 and the speed in Sect. C.3.3.

C.3.1. Experimental Settings:. We ran all experiments on a machine with 2.67GHz Intel Xeon E7-8837 CPUs and 1TB memory (up to 32GB was used). The real-world multigraphs used in our experiments are summarized in Table V. From them, we created fully-dynamic multigraph streams as in Sect. 5.1, while fixing α (i.e., the ratio of deleted edges) to 20%. We used TRIEST$_{\text{FD}}$, which supports triangle counting in fully-dynamic multigraph streams, as the competitor of THINKD.

C.3.2. Accuracy **(THINKD is more accurate than its best competitor)**. We compared the accuracies of THINKD$_{\text{FAST}}$, THINKD$_{\text{ACC}}$, and TRIEST$_{\text{FD}}$, as we changed the ratio of stored edges at the end of each input stream from 5% to 40%. Each evaluation metric was averaged over 1000 trials in all the datasets. As seen in Fig. 8, THINKD$_{\text{ACC}}$ consistently gave the best trade-off between space and accuracy, followed by THINKD$_{\text{FAST}}$. Specifically, within the same memory budget, THINKD$_{\text{ACC}}$ was up to $\mathbf{2.7\times}$ and $\mathbf{2.5\times}$ **more accurate** than TRIEST$_{\text{FD}}$ in terms of global error and RMSE, respectively.

Algorithm 4: Extensions of THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ to a Multigraph

```
Inputs  : fully dynamic multigraph stream: $(e^{(1)}, e^{(2)}, ...) = ((\{u,v\}, l, \delta), ...,)$,
          sampling probability: $r$ (for THINKD_FAST), and
          memory budget: $k$ $(\geq 2)$ (for THINKD_ACC)
Outputs: estimate of the global triangle count: $\bar{c}$
          estimates of the local triangle counts: $c[u]$ for each node $u$
1  $\mathcal{S} \leftarrow \emptyset$ (for THINKD_FAST)
2  $|\mathcal{E}| \leftarrow 0$, $n_b \leftarrow 0$, $n_g \leftarrow 0$ (for THINKD_ACC)
3  for each element $e^{(t)} = (\{u,v\}, l, \delta)$ in the input stream do
4      UPDATE($(\{u,v\}, l), \delta$)
5      if $\delta = +$ then INSERT($(\{u,v\}, l)$)
6      else if $\delta = -$ then DELETE($(\{u,v\}, l)$)
7  Procedure UPDATE($(\{u,v\}, l, \delta)$):          ▷ update global and local triangle counts
8      compute $\hat{\mathcal{N}}[u] \cap \hat{\mathcal{N}}[v]$          ▷ $\hat{\mathcal{N}}[u]$ and $\hat{\mathcal{N}}[v]$ are obtained from current $\mathcal{S}$
9      for each common neighbor $w \in \hat{\mathcal{N}}[u] \cap \hat{\mathcal{N}}[v]$ do
10         if THINKD_FAST then
11             if $\delta = +$ then increase $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $\frac{M(v,w)M(w,u)}{r^2}$
12             else if $\delta = -$ then decrease $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $\frac{M(v,w)M(w,u)}{r^2}$
13         else if THINKD_ACC then
14             if $\delta = +$ then increase $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $\frac{M(v,w)M(w,u)}{p(|\mathcal{E}|,n_b,n_g)}$
15             else if $\delta = -$ then decrease $\bar{c}$, $c[u]$, $c[v]$, and $c[w]$ by $\frac{M(v,w)M(w,u)}{p(|\mathcal{E}|,n_b,n_g)}$
16 Procedure INSERT($(\{u,v\}, l)$):
17     if THINKD_FAST then
18         if a random number in Bernoulli($r$) is 1 then $\mathcal{S} \leftarrow \mathcal{S} \cup \{(\{u,v\}, l)\}$
19     else if THINKD_ACC then
20         $|\mathcal{E}| \leftarrow |\mathcal{E}| + 1$
21         if $n_b + n_g = 0$ then
22             if $|\mathcal{S}| < k$ then $\mathcal{S} \leftarrow \mathcal{S} \cup \{(\{u,v\}, l)\}$
23             else if a random number in Bernoulli($k/|\mathcal{E}|$) is 1 then
24                 replace a random edge in $\mathcal{S}$ with $(\{u,v\}, l)$
25         else if a random number in Bernoulli($n_b/(n_b + n_g)$) is 1 then
26             $\mathcal{S} \leftarrow \mathcal{S} \cup \{(\{u,v\}, l)\}$, $n_b \leftarrow n_b - 1$
27         else $n_g \leftarrow n_g - 1$
28 Procedure DELETE($(\{u,v\}, l)$):
29     if THINKD_FAST then
30         if $(\{u,v\}, l) \in \mathcal{S}$ then $\mathcal{S} \leftarrow \mathcal{S} \setminus \{(\{u,v\}, l)\}$
31     else if THINKD_ACC then
32         $|\mathcal{E}| \leftarrow |\mathcal{E}| - 1$
33         if $(\{u,v\}, l) \in \mathcal{S}$ then
34             $\mathcal{S} \leftarrow \mathcal{S} \setminus \{(\{u,v\}, l)\}$, $n_b \leftarrow n_b + 1$
35         else $n_g \leftarrow n_g + 1$
```

C.3.3. Speed **(THINKD is faster than its best competitor)**. We compared the speeds and accuracies of THINKD$_{\text{FAST}}$, THINKD$_{\text{ACC}}$, and TRIEST$_{\text{FD}}$. The detailed settings were the same as those in Sect. C.3.2, and we ignored time taken to wait for the arrival of elements to measure the speeds of the algorithms independently of the speed of the input stream. As shown in Fig. 9, THINKD$_{\text{FAST}}$ consistently gave the best trade-off

Table V: Summary of the real-world multigraph streams used in our experiments. M: million, K: thousand.

Name	**#Nodes**	**#Edges**	**Type**
Enron [Klimt and Yang 2004]	87.0K	1.13M	Email Network
DBLP [Ley 2002]	1.28M	18.2M	Author Collaboration
Wikipedia [Sun et al. 2016]	2.86M	19.3M	Communication
Movie [Barabási and Albert 1999]	382K	33.1M	Actor Collaboration

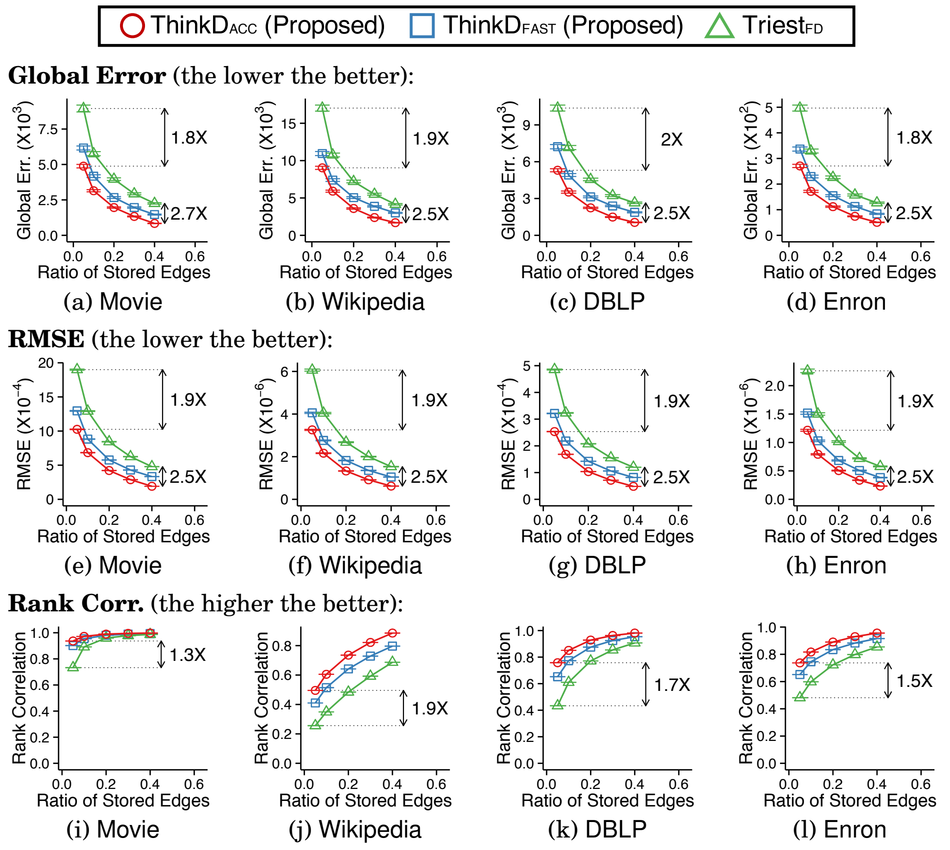


Fig. 8: **THINKD is accurate**. THINKD$_{\text{FAST}}$ and THINKD$_{\text{ACC}}$ provide a better trade-off between space and accuracy than TRIEST$_{\text{FD}}$. In particular, THINKD$_{\text{ACC}}$ is up to **2.7× more accurate** than TRIEST$_{\text{FD}}$ within the same memory budget.

between speed and accuracy, followed by THINKD$_{\text{ACC}}$. Specifically, for the same global error and RMSE, THINKD$_{\text{FAST}}$ was up to **2.2× faster** than TRIEST$_{\text{FD}}$.

D. APPLICATION: DETECTING SUDDENLY EMERGING DENSE SUBGRAPHS

In this section, we apply THINKD to the task of detecting dense subgraphs created within a short time. Specifically, we first provide a short survey on dense-subgraph detection and sliding-window models. Then, we propose an algorithm called THINKD-SPOT, which utilizes THINKD to detect suddenly emerging dense subgraphs. Lastly,

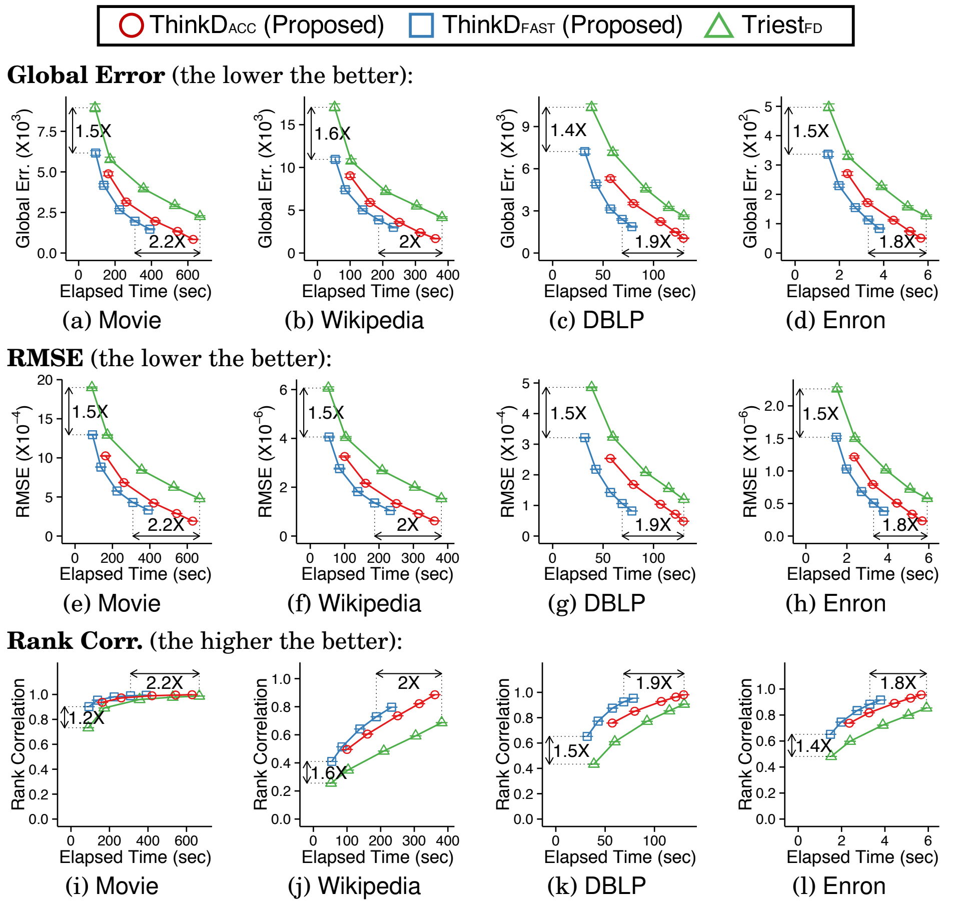


Fig. 9: **THINKD is fast**. THINKD$_{FAST}$ and THINKD$_{ACC}$ provide a better trade-off between speed and accuracy than TRIEST$_{FD}$. In particular, THINKD$_{FAST}$ is up to 2.2× **faster** than TRIEST$_{FD}$ when they are similarly accurate.

we experimentally show the advantages of THINKD-SPOT over state-of-the-art algorithms.

D.1. Survey on Detecting Dense Subgraphs and Sliding-Window Models

Detecting Dense Subgraphs. Theoretical work on identifying densest subgraphs includes exact and approximate algorithms for static graphs [Goldberg 1984; Charikar 2000; Andersen and Chellapilla 2009; Khuller and Saha 2009; Bahmani et al. 2012] and approximate algorithms for dynamic graphs [Epasto et al. 2015a; Bhattacharya et al. 2015; McGregor et al. 2015; Nasir et al. 2017]. We refer interested readers to a survey [Lee et al. 2010]. Unusually dense subgraphs in real-world graphs tend to signal anomalous or fraudulent behaviors, including a 'follower-boosting' service on social media [Jiang et al. 2014; Hooi et al. 2017; Shin et al. 2018a], 'copy-and-paste' bibliographies in citation networks [Prakash et al. 2010; Shin et al. 2018a], and websites that attempt to manipulate search engine rankings [Gibson et al. 2005]. Several recent algorithms detect dense subgraphs or more generally dense subtensors that are created within a short time or in a temporally synchronized manner [Maruhashi et al. 2011; Beutel et al. 2013; Jiang et al. 2015; Shin et al. 2017a; Shin et al. 2017b;

Rozenshtein et al. 2017; Shin et al. 2018c]. Considering temporal aspects enables these algorithms to accurately identify suspicious subgraphs and subtensors, which indicate fake 'Likes' on Facebook [Beutel et al. 2013], 'edit wars' and bot activities on Wikipedia [Shin et al. 2017b; Shin et al. 2018a], network intrusions [Maruhashi et al. 2011; Shin et al. 2017a], etc.

Sliding-Window Models. Our dense-subgraph detection method, described in the following section, is in *the timestamp-based sliding-window model* [Babcock et al. 2002], where the elements (or edges) whose arrival timestamp is within a time interval ΔT of the current time are maintained, while the others are "expired" and removed. This model is different from but similar to *the sequence-based sliding window model* [Datar et al. 2002], where the most recent k elements (or edges) are maintained, regardless of their arrival timestamps, while the others are removed. In addition to sampling algorithms in both models [Babcock et al. 2002], several graph algorithms in the latter model have been developed for connectivity tests, graph sparsification, minimum spanning forest detection, etc [Crouch et al. 2013; McGregor 2014]. Note that THINKD is in the fully-dynamic graph stream model, which is more general than both sliding-windows models. Designing triangle counting algorithm specific to the sliding-windows models remains to be investigated as future work.

D.2. Proposed Method: THINKD-SPOT

We present an algorithm called THINKD-SPOT, which utilizes THINKD to detect dense subgraphs that are created within a short time. THINKD-SPOT, which is described in Fig. 10, tracks the estimated counts of triangles using THINKD and uses the sudden increases in the counts as the signal of suddenly emerging dense subgraphs.

Consider a graph whose edges have actual timestamps (rather than logical timestamps), and for each edge e, let t_e be its timestamp. THINKD-SPOT first creates a fully dynamic graph stream with deletions as follows:

(1) For each edge e, create an addition $(e, +)$ with timestamp t_e,
(2) For each edge e, create a deletion $(e, -)$ with timestamp $t_e + \Delta T$,
(3) Sort the additions and deletions from the previous steps in the increasing order of their timestamps.

Then, THINKD-SPOT processes the created stream using THINKD (either THINKD$_{\text{FAST}}$ or THINKD$_{\text{ACC}}$). This makes THINKD-SPOT maintain the estimated counts of global and local triangles created within ΔT time units. THINKD-SPOT tracks the changes in the maintained estimate of the global triangle count, and if a spike with the estimate greater than a given threshold θ is encountered, THINKD-SPOT reports the emergence of a sudden dense subgraph. At each spike, THINKD-SPOT returns the maintained estimates of local triangle counts as the amount of contribution of each node to the sudden dense subgraph.

Although we describe THINKD-SPOT in an offline setting, it can be easily extended to online settings where new edges are added to the input graph while it is being processed. In online settings, to maintain the estimated counts of triangles created within ΔT time units, THINKD-SPOT processes the insertion of each new edge as soon as the edge arrives, while it processes the deletion of the edge after ΔT time units. While new edges need to be stored until they are removed after ΔT units, since they are written and deleted sequentially, they can be stored on disks.

D.3. Experiments

For empirical evaluation of THINKD-SPOT, we measured how rapidly and accurately THINKD-SPOT detects suddenly emerging dense subgraphs compared to its best competitors: DENSEALERT [Shin et al. 2017b], M-BIZ [Shin et al. 2018c], D-CUBE [Shin

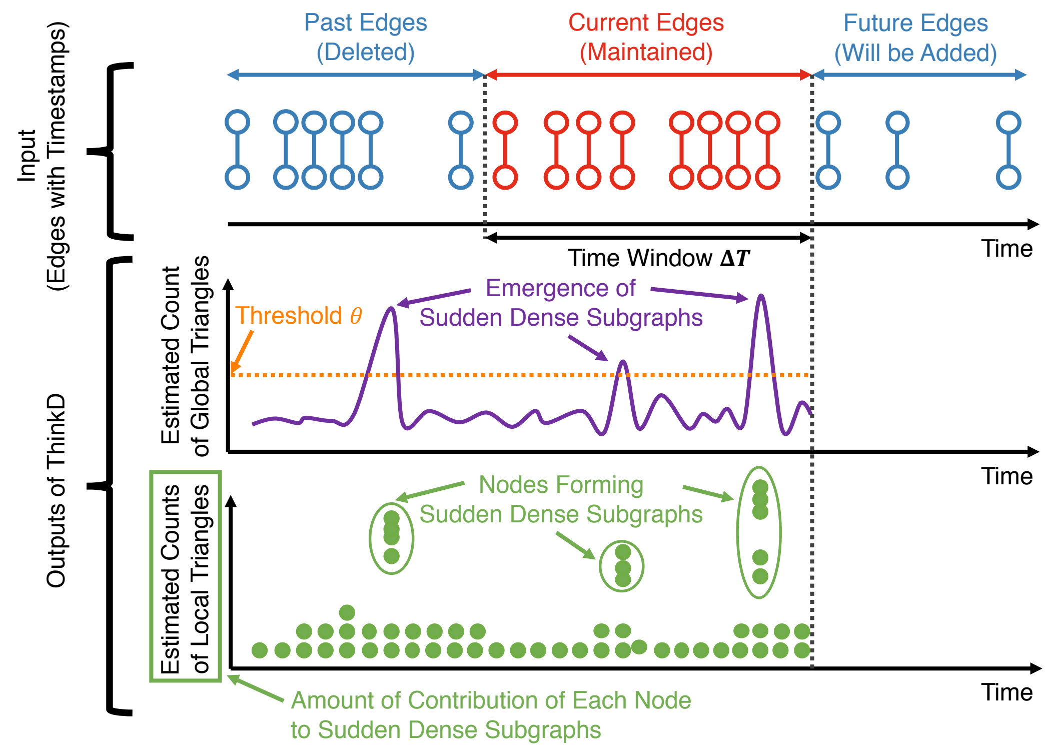

Input
(Edges with Timestamps)
Past Edges
(Deleted)
Current Edges
(Maintained)
Future Edges
(Will be Added)
Time
Time Window ΔT
Outputs of ThinkD
Estimated Count of Global Triangles
Threshold θ
Emergence of Sudden Dense Subgraphs
Time
Estimated Counts of Local Triangles
Nodes Forming Sudden Dense Subgraphs
Time
Amount of Contribution of Each Node to Sudden Dense Subgraphs

Fig. 10: Pictorial description of THINKD-SPOT, which spots suddenly emerging dense subgraphs using THINKD. THINKD-SPOT maintains the estimated counts of global and local triangles created within ΔT time units. Spikes in the estimated count of global triangles indicate the emergence of sudden dense subgraphs, and the estimated counts of local triangles indicate the amount of contribution of each node to the sudden dense subgraphs.

et al. 2017a], and CROSSSPOT [Jiang et al. 2015], all of which are designed for detecting dense subgraphs or more generally dense subtensors created within a short time.[12] Specifically, we randomly injected 10 cliques created within a day into the Facebook and Epinion datasets.[13] That is, the timestamps of all edges composing each clique were within a day. The injected cliques were of size 6 to 15. Then, we measured the running time of each method and the precision at the top-10 outputs obtained by each method (i.e., the ratio of outputs that correspond to the injected cliques).

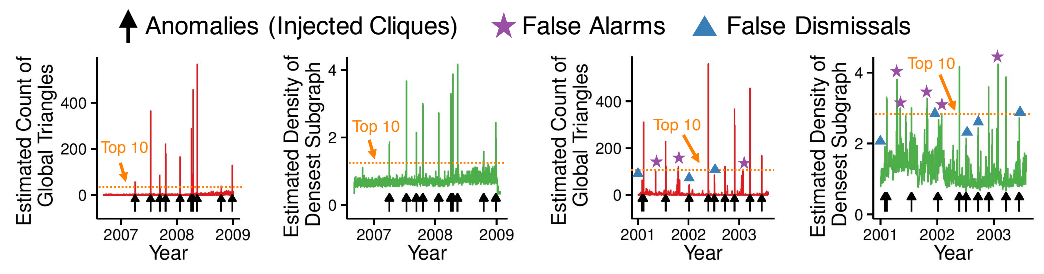
As summarized in Fig. 11(a), THINKD-SPOT was the fastest and the most accurate in both datasets. Especially, THINKD-SPOT was $8.3 - 19.6\times$ **faster** than DENSEALERT, which was the second most accurate method. Figs. 11(b)-(e) show that the spikes in the estimated count of global triangles obtained by THINKD-SPOT indicated the in-

[12]We implemented all the algorithms commonly in Java. We used THINKD$_{\text{ACC}}$ with $k = 2000$ for triangle counting in THINKD-SPOT, and we set ΔT to a day for both THINKD-SPOT and DENSEALERT. We used the geometric average degree as the density measure in M-BIZ and D-CUBE, since this measure led to the highest accuracy. Using each of M-BIZ, D-CUBE, and CROSSSPOT, we detected 10 dense subgraphs. The other experimental settings were the same as those described in Sect. 5.1.

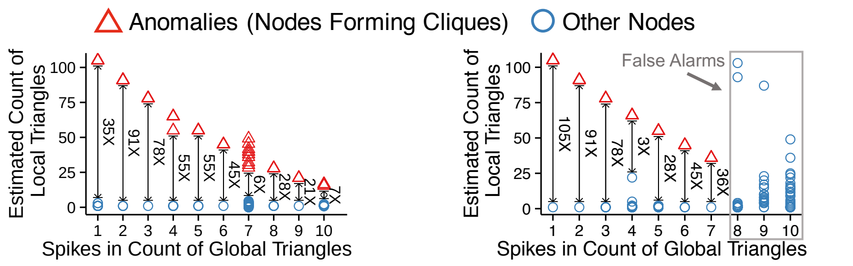
[13]We used the Facebook and Epinion datasets because the edges in them have timestamps. We ignored the edges with missing timestamps.

Datasets	Facebook		Epinion	
Measures	Running Time (milliseconds)	Precision @ Top-10	Running Time (milliseconds)	Precision @ Top-10
CROSSSPOT	54,975	0.00	11,173	0.00
D-CUBE	1,443	0.33	674	0.00
M-BIZ	1,298	0.31	667	0.41
DENSEALERT	12,080	**1.00**	1,843	0.59
THINKD-SPOT	**617**	**1.00**	**223**	**0.75**

(a) Running Time and Accuracy Averaged over 10 Runs.



(b) Spikes from THINKD-SPOT (Facebook) (c) Spikes from DENSEALERT (Facebook) (d) Spikes from THINKD-SPOT (Epinion) (e) Spikes from DENSEALERT (Epinion)



(f) Local Triangle Counts at Top-10 Spikes (Facebook) (g) [Local Triangle Counts at Top-10 Spikes (Epinion)

Fig. 11: (a) **THINKD-SPOT is fast and accurate**. (b-e) The spikes in the estimated count of global triangles, obtained by THINKD-SPOT, indicate the injected suddenly emerging cliques more accurately than the spikes obtained by DENSEALERT. The arrows indicate the timestamps where the cliques are injected, and the stars indicate the false alarms in top-10 spikes. The triangles indicate false dismissals. (f-g) The nodes forming the injected cliques are clearly distinguished by their estimated counts of local triangles, obtained by THINKD-SPOT. The plots show the distributions of the estimated counts of local triangles at the top-10 spikes in (b-e).

jected cliques more accurately than the spikes obtained by DENSEALERT. Figs. 11(e)-(f) show that the nodes forming the injected cliques are clearly distinguished by their estimated counts of local triangles, obtained by THINKD-SPOT.

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